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On Perfect 2-Coloring of the Bicubic Graphs with Order up to 12

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Abstract.: A perfect coloring of a graph G with m color (a perfect m-coloring) is a surjective mapping $P : V(G) \rightarrow \{1, 2, \ldots, m\}$ such that each vertex of color i has exactly m_{ij} neighbors of color j, for all i, j, where $M = (m_{ij})_{i,j=1,2,\ldots,m}$ is the corresponding matrix. In this paper, we classify perfect 2-colorings of the bicubic graphs with order up to 12.

AMS (MOS) Subject Classification Codes: 05c15; 05c17
Key Words: Bicubic graphs; vertex perfect coloring; equitable partition.

1. Introduction

Perfect coloring of graphs with m colors is a new field in mathematics that is connected to algebra, graph theory and combinatorics [12]. Completely regular codes in graphs (existence part of it is the result of a question and historical issue in mathematics) are generalizations of perfect codes [11]. This problem started with Delsarte's conjecture (Johnson's graphs lack perfect codes) [6]. Delsarte's conjecture is the basis of a conceptual research dealing with perfect coloring of graphs [7, 11]. Assume $E^n = \Big\{(x_1, x_2, \ldots, x_n); x_i \in \{0,1\}\Big\}$, where $n \in \mathbb{N}$. For each $x \in E^n$, we define $W(x)$ the weight x, as the number of non-zero components corresponding to x. The vertex set of the Johnson graph $J(n, \omega)$ contains vectors with weight ω in E^n. Two vertices of Johnson graph are adjacent if and only if their corresponding vectors have exactly two different components [3, 11, 14]. Furthermore, perfect coloring of some Johnson graphs including $J(\mathrm{v}, 3)$ (v odd), $J(8, 4)$, $J(8, 3)$, $J(6, 3)$ and other graphs such as the Hypercube graphs, the generalized Petersen graphs and cubic graphs have been settled [1, 3, 4, 7, 10, 11, 14, 16]. Fon-Der-Flass calculated and counted the parameter matrices of the n-dimensional cube with a given parameter matrix and furthermore he got some structures for the existence of perfect 2-colorings of the n-dimensional cube [8, 9, 10].

From now on, we denote by p2-c the perfect coloring of a graph with 2 colors.

The aim of this paper is to classify all parameter matrices of p2-c of the bicubic graphs with order up to 12.

2. Definitions and Preliminaries

Some basic definitions used in this paper are given in this section. Let $G = (V, E)$ be a connected graph without loops or multiple edges. A bicubic graph is a bipartite 3-regular (cubic) graph.

Definition 2.1. [13, Section 9.3] *Equitable partition of $G = (V, E)$ graph with m parts, is a partition of V with parts of $Q_1, Q_2, \ldots, Q_m$ such that for $i, j \in \{1, 2, 3, \ldots, m\}$ there is a nonnegative integer $h_{i,j}$ such that each vertex v in Q_i has exactly $h_{i,j}$ neighbors in Q_j, regardless of the choice of v. The partition matrix is $H = (h_{i,j})$.*

Definition 2.2. [2, Definition 2.1] *A perfect coloring of a graph G with m colors (a perfect m-coloring) with matrix $M = (m_{ij})_{i,j=1,2,\ldots,m}$ is a mapping $P : V(G) \to \{1, 2, \ldots, m\}$ such that P is surjective, and for all i, j, for every vertex of color i, the number of its neighbors of color j is equal to $m_{i,j}$. The matrix M is called the parameter matrix of a perfect coloring.*

If $m = 2$, then the first color is considered white and the second one is considered black.

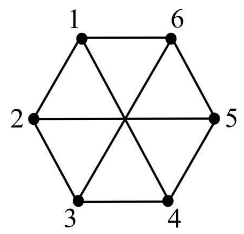
Remark 2.3. [15] *The connected bicubic graphs of orders 6 to 12 are divided into four classes based on their number of vertices. This classification is shown in Figures 1 to 4.*

The next lemma calculates the number of white vertices in a perfect 2-coloring.

Lemma 2.4. [3, Proposition 1] *If W is the set of white vertices in a p2-c of a graph $G = (V, E)$ with parameter matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then*

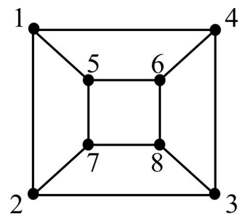
$$|W| = |V(G)| \tfrac{c}{c+b}.$$

Remark 2.5. [3, Section 1] *Suppose $G = (V, E)$ is a connected k-regular graph. Then the first condition for existence of a perfect coloring with two colors of G with parameter matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the following equality:*



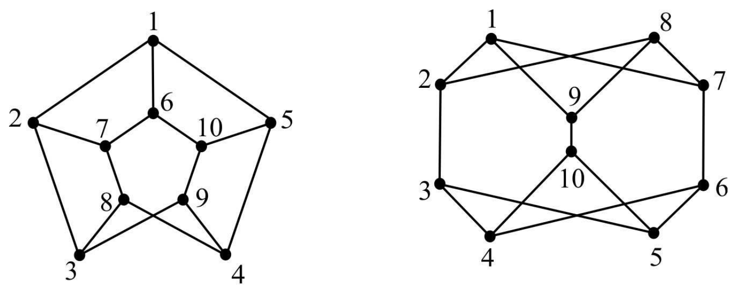
Bc1

FIGURE 1. Connected bicubic graphs of order 6 (Bc1).



Bc2

FIGURE 2. Connected bicubic graphs of order 8 (Bc2).



Bc3 Bc4

FIGURE 3. Connected bicubic graphs of order 10 (Bc3 and Bc4).

$$a + b = c + d = k.$$

The second condition is obtained from the connectivity of G as follows:

$$b, c \neq 0.$$

Lemma 2.6. [12, Lemma 1.1] *If P is a perfect m-coloring of a graph $G = (V, E)$, then P and G have the same eigenvalues.*

Lemma 2.7. [2, Corollary 2.4] *Let P be a p2-c with parameter matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of a k-regular graph G. Then the numbers $a - c$ and k are eigenvalues of P and so of G.*

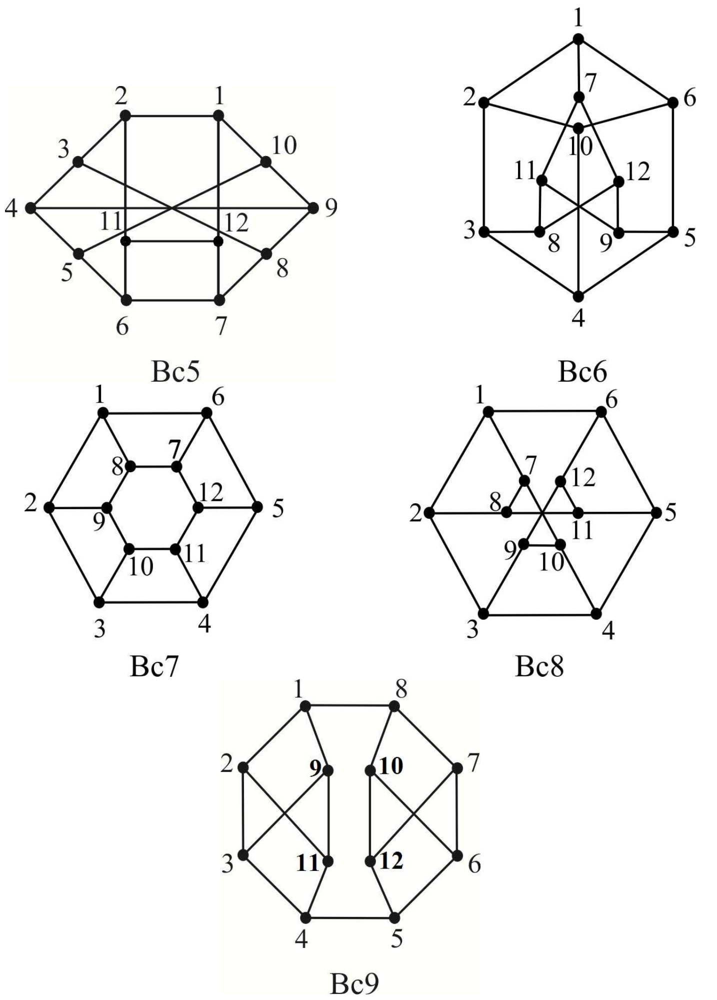


FIGURE 4. Connected bicubic graphs of order 12 (Bc5, Bc6, Bc7, Bc8 and Bc9).

3. PERFECT 2-COLORING OF THE BICUBIC GRAPHS

In this section, the corresponding parameter matrices related to the p2-c bicubic graphs of order up to 12 will be calculated.

Lemma 3.1. *Let $G = (V, E)$ be a connected bicubic graph. Then, the following six matrices are the only parameter matrices of a p2-c P of G:*

$$M_1 = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, M_5 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, M_6 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Proof. By Remark 2.5, Lemma 2.6 and Lemma 2.7 we have the result. □

In the following theorem, we will show that all the bicubic graphs have p2-c with parameter matrix M_1.

Theorem 3.2. *All the bicubic graphs with orders less than 12 have a p2-c with parameter matrix M_1.*

Proof. To prove, we define p2-c for the bicubic graphs of orders 6 to 12 with matrix M_1 as follows:
p2-c P for Bc1 with matrix M_1
$P(6) = P(4) = P(2) = 2, \quad P(5) = P(3) = P(1) = 1.$
p2-c P for Bc3 with matrix M_1
$P(10) = P(7) = P(4) = P(3) = P(1) = 1, \quad P(9) = P(8) = P(6) = P(5) = P(2) = 2.$
p2-c P for Bc4 with matrix M_1
$P(10) = P(8) = P(6) = P(3) = P(1) = 1, \quad P(9) = P(7) = P(5) = P(4) = P(2) = 2.$
p2-c P for Bc5 with matrix M_1
$P(i) = 1$, for each even number i, $P(j) = 2$ for each odd number j.
p2-c P for Bc6 with matrix M_1
$P(12) = P(11) = P(10) = P(5) = P(3) = P(1) = 1$, and for other vertices i, we define $P(j) = 2$.
p2-c P for Bc7 with matrix M_1
$= P(11) = P(9) = P(7) = P(5) = P(3) = P(1) = 1$, and for other vertices i, we define $P(j) = 2$.
p2-c P for Bc8 with matrix M_1
$P(12) = P(10) = P(8) = P(5) = P(3) = P(1) = 1$, and for rest of the vertices are black.
p2-c P for Bc9 with matrix M_1
$P(11) = P(10) = P(7) = P(5) = P(3) = P(1) = 1$, and we define the color of the rest of the vertices as black.

It is easily seen that above functions are p2-c with parameter matrix M_1. □

Lemma 3.3. [1] *The parameter matrices of p2-c of $Bc2$ are M_1, M_3, M_4 and M_6.*

In the following theorem, we will investigate the parameter matrix M_3:

Theorem 3.4. *Except for $Bc2$, none of the other p2-c of the bicubic graph with orders less than 12 has M_3 as a parameter matrix.*

Proof. In Lemma 3.3, it was shown that the graph Bc2 has a p2-c with M_3. From Lemma 2.4, we conclude that there is no p2-c of graphs $Bc1, Bc3$ and $Bc4$ with matrix M_3. We next show that other bicubic graphs have no p2-c with M_3.

Suppose there is a p2-c P of Bc5 with parameter matrix M_3. From M_3, Without any loss of generality, we can assume that $P(1) = 1$ and then $P(2) = P(10) = P(12) = 2$.

Thus, we obtain $P(3) = P(11) = 2$ and $P(6) = 1$. Therefore, $P(5) = P(7) = P(8) = 2$ and $P(4) = 1$. From this we get $P(9) = 2$, which leads to a contradiction. Therefore the graph Bc5 has no p2-c with matrix M_3. For other graphs $Bc6, Bc7, Bc8$ and $Bc9$, we can get the same result. □

In the next theorem, it will be prove that the graph $Bc4$ has p2-c with parameter matrix M_1 and M_2.

Theorem 3.5. *The parameter matrices of p2-c of graph $Bc4$ are M_1 and M_2.*

Proof. In Theorem 3.2, it was shown that the graph $Bc4$ has a p2-c with M_1. Also the mapping defined by:

$$P(8) = P(5) = P(4) = P(1) = 1,$$
$$P(10) = P(9) = P(7) = P(6) = P(3) = P(2) = 2,$$

gives a p2-c of $Bc4$ with matrix M_2. Using Lemma 2.4 and Theorem 3.4, we conclude that there is no p2-c of graph $Bc4$ with matrices M_5 and M_3. Also, there is no p2-c for graph $Bc4$ with matrices M_4 and M_6. Suppose there is a p2-c of $Bc4$ with M_4. Then each vertex with color 1 has one adjacent vertex with color 1. We have the following cases:

(1) $P(1) = P(2) = 1;$

(2) $P(2) = P(3) = 1;$

(3) $P(4) = P(6) = 1;$

(4) $P(9) = P(10) = 1.$

In case (1), the vertex by color 2 has two adjacent vertices with color 2, which is a contradiction with the second row of M_4, and for other cases we have the same results. By the same proof, for parameter matrix M_6, each vertex with color 1 has two adjacent vertices with color 1. We have three cases:

(1) $P(1) = P(2) = 1$ and $P(7) = 1;$

(2) $P(1) = P(2) = 1$ and $P(9) = 1;$

(3) $P(9) = P(8) = 1$ and $P(10) = 1.$

In all cases, the vertex with color 1, has two adjacent vertices with color 1, which is a contradiction with the second row of M_6. Therefore graph $Bc4$ has no p2-c with matrices M_4 and M_6. □

In the next theorem, it will be prove that the graph $Bc4$ is the only one that has a p2-c with parameter matrix M_2.

Theorem 3.6. *Except for $Bc4$, none of the other p2-c of the bicubic graph with orders less than 12 has M_2 as a parameter matrix.*

Proof. In Theorem 3.5, it was shown that the graph $Bc4$ has a p2-c with M_2. From Lemma 2.4, we conclude that there is no p2-c of graphs $Bc1, Bc2, Bc5, Bc6, Bc7, Bc8$ and $Bc9$ with matrix M_2. We next show that the graph $Bc3$ has no p2-c with matrix M_2.

Suppose there is a p2-c P of $Bc3$ with parameter matrix M_2. From M_2, Without any loss of generality, we can assume that $P(1) = 1$ and then $P(2) = P(5) = P(6) = 2$.

Thus, we obtain $P(7) = P(10) = 1$. Therefore, $P(8) = P(9) = 2$. From this we get $P(3) = P(4) = 2$, which leads to a contradiction with the second row of M_2 . Therefore the graph Bc3 has no p2-c with matrix M_2. □

Finally, we can list all the parameter matrices of the bicubic graphs with orders 6 to 12 in the next theorem:

Theorem 3.7. *The parameter matrices of a p*2*-c of the bicubic graphs with orders* 6 *to* 12 *are illustrated in the following table:*

TABLE 1. Parameter matrices of the bicubic graphs.

graph matrices	matrix M_1	matrix M_2	matrix M_3	matrix M_4	matrix M_5	matrix M_6
Bc1	√				√	
Bc2	√		√	√		√
Bc3	√					
Bc4	√	√				
Bc5	√			√	√	
Bc6	√				√	
Bc7	√			√	√	√
Bc8	√			√		√
Bc9	√				√	

Proof. From Lemma 3.3, we deduce that the graph $Bc2$ has a p2-c with matrices M_1, M_3, M_4 and M_6. Also, in Theorem 3.5, we showed that the graph $Bc4$ has perfect 2-coloring with parameter matrices M_1 and M_2, then in Theorem 3.4 we obtained that there are no p2-c of bicubic graphs with orders 6 to 12 with matrix M_3, except graph $Bc2$ and in Theorem 3.6, we showed that the only graph $Bc4$ has p2-c with parameter matrix M_2. Now, we study other parameter matrices and graphs listed in Table 1. First, we define a p2-c P for possible cases listed in Table 1 as follow:

p2**-c** P **for Bc1 with matrix** M_5

$P(6) = P(1) = 1, \quad P(5) = P(4) = P(3) = P(2) = 2.$

p2**-c** P **for Bc5 with matrix** M_4

$P(9) = P(7) = P(6) = P(4) = P(2) = P(1) = 1,$

$P(12) = P(11) = P(10) = P(8) = P(5) = P(3) = 2.$

p2**-c** P **for Bc5 with matrix** M_5

$P(12) = P(11) = P(9) = P(4) = 1,$

$P(10) = P(8) = P(7) = P(6) = P(5) = P(3) = P(2) = P(1) = 2.$

p2**-c** P **for Bc6 with matrix** M_5

$P(12) = P(10) = P(7) = P(4) = 1,$

$P(11) = P(9) = P(8) = P(6) = P(5) = P(3) = P(2) = P(1) = 2.$

p2**-c** P **for Bc7 with matrix** M_4

$P(12) = P(10) = P(8) = P(5) = P(3) = P(1) = 1,$

$P(11) = P(9) = P(7) = P(6) = P(4) = P(2) = 2.$

p2**-c** P **for Bc7 with matrix** M_5

$P(10) = P(7) = P(6) = P(3) = 1$,
$P(12) = P(11) = P(9) = P(8) = P(5) = P(4) = P(2) = P(1) = 2$.
p2-c P for Bc7 with matrix M_6
$P(6) = P(5) = P(4) = P(3) = P(2) = P(1) = 1$,
$P(12) = P(11) = P(10) = P(9) = P(8) = P(7) = 2$.
p2-c P for Bc8 with matrix M_4
$P(12) = P(8) = P(7) = P(6) = P(4) = P(3) = 1$,
$P(11) = P(10) = P(9) = P(5) = P(2) = P(1) = 2$.
p2-c P for Bc8 with matrix M_6
$P(6) = P(5) = P(4) = P(3) = P(2) = P(1) = 1$,
$P(12) = P(11) = P(10) = P(9) = P(8) = P(7) = 2$.
p2-c P for Bc9 with matrix M_5
$P(10) = P(8) = P(4) = P(3) = 1$,
$P(12) = P(11) = P(9) = P(7) = P(6) = P(5) = P(2) = P(1) = 2$.

It is obvious that above functions are p2-c with their mentioned parameter matrices. Now, we prove that there are no p2-c for other graphs listed in Table 1. For example, there is no p2-c of $Bc3$ with the matrix M_6. Otherwise, to obtain a contradiction, assume that there is a p2-c of $Bc3$ with the parameter matrix M_6. Without any loss of generality, we can assume that $P(1) = P(2) = P(6) = 1$. From M_6 we have $P(5) = P(4) = P(9) = P(10) = 2$. Thus we conclude that $P(3) = 1$ and $P(8) = 2$. This is a contradiction with $m_{21} = 1$. Therefore, the graph $Bc3$ has no p2-c with matrix M_6. For other graphs in Table 1, one can give a similar proof. □

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