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Further Results on Intersection Power Graph of Finite Groups

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Abstract.: Let G be a finite group with identity element e. The intersection power graph $\Gamma_{IP}(G)$ of G is the undirected graph whose vertex set is the elements of G and two distinct vertices a, b are adjacent in $\Gamma_{IP}(G)$ if there exists a non-identity element c such that $a^p = c = b^q$ for two positive integers p, q and e is adjacent to all other vertices of $\Gamma_{IP}(G)$. In the present paper, we determine some finite groups whose intersection power graphs having book thickness at most two. Also, we attain a lower bound for $\alpha_0(\Gamma_{IP}(G))$.

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1. Introduction

All groups considered in this paper will be finite. The *undirected power graph* of a finite group G has the vertex set G and two distinct elements are adjacent if one is a power of the other. In 2000, Kelarev and Quinn [8] introduced the concept of a power graph. In 2009, Chakrabarty, Ghosh and Sen [6] introduced the concept of undirected power graph. Recently, many interesting results on power graphs have been obtained, see [1, 5, 6, 9, 10].

A detailed list of results and open questions on power graphs can be found in [2]. Then in 2018, Bera [3] introduced the intersection power graph of a finite group. Sudip Bera [3] defined the intersection power graph $\Gamma_{IP}(G)$ of a finite group G as follows: take the group elements as the vertex set and two distinct vertices a, b are adjacent in $\Gamma_{IP}(G)$ if $\langle a\rangle \cap \langle b\rangle \neq \{e\}$ and e is adjacent to all other vertices of $\Gamma_{IP}(G)$, where e is the identity element of G.

We recall some graph terminology. A graph in which each pair of distinct vertices is joined by the edge is called a *complete graph*. We use K_n to denote the complete graph with n vertices. If the induced subgraph on a subset A of the vertices of a graph $\mathcal{G}$ has no edges, then A is said to be an *independent set* [7] of $\mathcal{G}$. The largest size of an independent set of a graph $\mathcal{G}$ is the *independence number* of $\mathcal{G}$ [7]. Let us denote it as $\alpha_0(\mathcal{G})$. A connected graph with only one cycle is called a *unicyclic graph*. A graph without edges is called a *null graph*. The largest distance between two vertices of a graph is said to be the *diameter* of the graph. An undirected planar graph is said to be a *friendship graph* if it has $2n+1$ vertices and $3n$ edges, and let us denote it as F_n. One can construct the friendship graph by joining n copies of C_3 at a common vertex. A book is a collection of half-planes all having the same line as their boundary. A planar embedding of a graph into a book is called the book embedding. The smallest possible number of half-planes for any book embedding of a graph is called the *book thickness* of the graph $\mathcal{G}$. Let us denote it as $bt(\mathcal{G})$.

In this paper, we address problems concerning unicyclicity and some bounds for the diametre, the book thickness and the independent number of the intersection power graph in some especial cases. The motivation for these results comes from [1] and [4], where alike results for some graphs related to finite groups have been addressed.

2. Properties of $\Gamma_{IP}(G)$

In the present section, we provide some basic properties of $\Gamma_{IP}(G)$.

Theorem 2.1. *Let G be a finite group. Then $\Gamma_{IP}(G)$ is unicyclic if and only if $G \cong \mathbb{Z}_3$ or S_3, where S_3 is the symmetric group on 3 letters.*

Proof. Clearly, $\Gamma_{IP}(\mathbb{Z}_3)$ is a cycle of length 3. $\Gamma_{IP}(S_3)$ has precisely one cycle of length 3 induced by the identity element and two elements of order 3.

Conversely, suppose that $\Gamma_{IP}(G)$ is unicyclic. Then we may prove the following:
1) $|G|$ has no prime divisor p with $p \geq 5$ since $\Gamma_{IP}(G)$ is unicyclic. As a result, we have $|G| = 2^m3^n$.
2) Consider a Sylow 3-subgroup M of G. Suppose that $|M| \geq 9$. If M has an element x of order 9, then in $\langle x\rangle$, $\Gamma_{IP}(G)$ has at least two cycles, a contradiction. If M has no elements of order 9, then every non-trivial element of M has order 3, which also implies that $\Gamma_{IP}(G)$ has at least two cycles, a contradiction.
3) By 2), we have $|G| = 2^m3$ and G has a unique Sylow 3-subgroup.
4) Similarly, we can conclude that G has no elements of order 4 and 6.
5) Let $P = \langle x\rangle$ is the unique Sylow 3-subgroup of G. Then P is normal in G. By 'N/C' Theorem, $G/C_G(P)$ can be imbedded in $Aut(P)$. Suppose there exists an element y in $G\backslash P$ such that $xy = yx$. Then the order of y is 2. Since the order of x and the order of y are coprime, one has that the order of xy is 6, which is impossible. It means that $C_G(P) = P$. Also, it is well known that $Aut(P) \cong \mathbb{Z}_2$. Hence, $|G/P| = 1$ or 2. In the former case one

has $G = P = Z_3$, as desired. In the latter case one has $|G| = 6$, one can easily verify that $G = S_3$ $\square$

Theorem 2.2. *For a finite group G with order $n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_m^{\beta_m}$, where $p_1, p_2, \ldots, p_m$ are different primes and $\beta_1, \beta_2, \ldots, \beta_m$ are natural numbers, G has unique subgroups with orders $p_1, p_2, \ldots, p_m$, u and v are non-adjacent in $\Gamma_{IP}(G)$, where $u, v(\neq e) \in G \Leftrightarrow$ the greatest common divisor between the order of u and the order of v is one.*

Proof. Suppose that the greatest common divisor between the order of u and the order of v is one, from this $\langle u\rangle \cap \langle v\rangle = \{e\}$. Therefore u and v are not adjacent.

Conversely, suppose that u and v are not adjacent. If the greatest common divisor between the order of u and the order of v is not equal to one, then there exists p_i such that p_i divides $o(u)$ and p_i divides $o(v)$. Which implies $\langle u\rangle$ and $\langle v\rangle$ should contain a subgroup with order p_i. Since the group has a unique subgroup with order p_i, $|\langle u\rangle \cap \langle v\rangle| \geq p_i$. Therefore u is adjacent to v, this is a contradiction. Hence the greatest common divisor between the order of u and the order of v is one $\square$

Theorem 2.3. *For a finite group G whose order is $p_1^{\beta_1} p_2^{\beta_2} \cdots p_m^{\beta_m}$, where $p_1, p_2, \ldots, p_m$ are distict primes and $\beta_1, \beta_2, \ldots, \beta_m$ are natural numbers, $\Gamma_{IP}(G)$ is connected and the diameter of $\Gamma_{IP}(G)$ is lessthan or equal to 4 if G contains unique subgroups whose orders are $p_1, p_2, \ldots, p_m$.*

Proof. Since p_i is a divisor of $|G|$, where $i = 1, 2, \ldots, m$, there exists $x_i \in G$ such that the order of x_i is p_i, where $i = 1, 2, \ldots, m$. Let x_i and x_j be two elements in G of order p_i and p_j, where $i \neq j$ and $1 \leq i, j \leq m$. Let $N_i = \langle x_i\rangle$ and $N_j = \langle x_j\rangle$ be subgroups of G. From our assumptions, N_i and N_j are unique subgroups with order p_i and p_j. From this N_i and N_j are normal subgroups of G. Also, N_iN_j is a normal subgroup of G so that the order of N_iN_j is p_ip_j. Since N_i and N_j are cyclic subgroups, N_iN_j is a cyclic subgroup so that it has an element y with order p_ip_j. From our assumptions, $\langle x_i\rangle \cap \langle y\rangle = \langle x_i\rangle$ and $\langle x_j\rangle \cap \langle y\rangle = \langle x_j\rangle$. Which implies that x_iyx_j is a path in $\Gamma_{IP}(G)$. Let u, v be two elements in G. Then there is p_i and p_j, for certain i, j and $1 \leq i, j \leq m$ such that p_i divides the order of u and p_j divides the order of v. Note that ux_iyx_jv is a path between u and v in $\Gamma_{IP}(G)$. Hence $\Gamma_{IP}(G)$ is connected and $diam(\Gamma_{IP}(G)) \leq 4$ $\square$

Theorem 2.4. [7] *K_5 and $K_{3,3}$ are non-planar.*

Example 2.5. *For an abelian group G with order either 12 or 18, $\Gamma_{IP}(G)$ is non-planar.*

Proof. Case 1: G is a cyclic group.
For $|G| = 12$, $G \cong \mathbb{Z}_{12}$. Since G has four elements with order 12 also a unique subgroup with order 2, K_5 is a subgraph of $\Gamma_{IP}(G)$. By Theorem 2.4, $\Gamma_{IP}(G)$ is non-planar.
For $|G| = 18$, $G \cong \mathbb{Z}_{18}$. Since G has six elements with order 18, K_6 is a subgraph of $\Gamma_{IP}(G)$. By Theorem 2.4, $\Gamma_{IP}(G)$ is non-planar.
Case 2: G is a non-cyclic group.
For $|G| = 12$, $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$. Since G has six elements with order 6, K_6 is a subgraph of $\Gamma_{IP}(G)$. By Theorem 2.4, $\Gamma_{IP}(G)$ is non-planar.
For $G = 18$, $G \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Since G has eight elements with order 6, K_8 is a subgraph of $\Gamma_{IP}(G)$. By Theorem 2.4, $\Gamma_{IP}(G)$ is non-planar $\square$

Lemma 2.6. *For any two finite groups H_1 and H_2, $\Gamma_{IP}(H_1) \cong \Gamma_{IP}(H_2)$ if $H_1 \cong H_2$.*

Proof. Suppose that $f : H_1 \to H_2$ is a group isomorphism. Let a, $b \in H_1$ such that a is adjacent to b in $\Gamma_{IP}(H_1)$. Since $\langle a\rangle \cong \langle f(a)\rangle$, for each $a \in H_1$, $|\langle a\rangle \cap \langle b\rangle| = |\langle f(a)\rangle \cap \langle f(b)\rangle| \geq 1$. From this $f(a)$ is adjacent to $f(b)$ in H_2. Therefore $\Gamma_{IP}(H_1) \cong \Gamma_{IP}(H_2)$ $\square$

Remark 2.7. *The converse of Lemma 2.6 is not true. Let us consider $(\mathbb{Z}_8, +_8)$ and the quaternion group Q_8 whose order is eight. Note that $\mathbb{Z}_8$ is not isomorphic to Q_8, but $\Gamma_{IP}(\mathbb{Z}_8) \cong K_8 \cong \Gamma_{IP}(Q_8)$.*

Theorem 2.8. *For any dihedral group G, $\Gamma_{IP}(G)$ is a tree $\Leftrightarrow G \cong D_2$ or D_4.*

Proof. If G is isomorphic to either D_2 or D_4, then the itersection power graph of G is either K_2 or $K_{1,3}$. Hence $\Gamma_{IP}(G)$ is a tree.

Conversely, suppose that $\Gamma_{IP}(G)$ is a tree. Suppose there is a prime number $p \geq 5$ such that p is a divisor of $|G|$. Since G has an element with order p, $\Gamma_{IP}(G)$ contains $K_p (p \geq 5)$ as a subgraph. From this $\Gamma_{IP}(G)$ is not a tree, a contradiction. Therefore $|G| = 2^n 3^m$, where $n \geq 1$ and $m \geq 0$ are two integers.
Suppose $|G| = 2^n 3^m$, where $n \geq 3$ and $m = 0$. Then $\Gamma_{IP}(G)$ contains $K_{2^{n-1}}$ as a subgraph. From this $\Gamma_{IP}(G)$ is not a tree, a contradiction.
Suppose $|G| = 2^n 3^m$, where $n \geq 1$ and $m \geq 1$. which implies G has an element w with order 3. Now, the subgraph induced by $\langle w\rangle$ contains K_3. From this $\Gamma_{IP}(G)$ contains K_3 as a subgraph, a contradiction. This implies that G is isomorphic to either D_2 or D_4 $\square$

3. Book thickness of $\Gamma_{IP}(G)$

In the present section, some finite groups whose $\Gamma_{IP}(G)$ having book thickness at most two are classified.

Theorem 3.1. [4] *For $m \geq 4$, $bt(K_m) = \lceil \frac{m}{2} \rceil$.*

Theorem 3.2. *For a dihedral group G, $bt(\Gamma_{IP}(G))$ is at most two if $G \cong D_{2n}$, where $n = 1, 2, 3, 4$.*

Proof. Since the intersection power graph of D_2, D_4 and D_6 have at least one edge, the book thickness for these graphs are at least one. Here each is a subgraph of the one page embeddable graph (see figure 3.1) for some integer n. Hence the book thickness for these graphs are one.

1 2 3 4 .... n a

Figure 3.1: One page embeddable graph.

Since the intersection power graph of D_8 contains K_4 as a subgraph and so it follows from Theorem 3.1 that, the book thickness for this graph is at least two.
In figure 3.2, two page embedding of the intersection power graph of G is portrayed. The first page lies above the spine and the second page lies below the spine. Hence the book thickness for this graph is two.

1 r r^2 r^3 s sr sr^2 sr^3

Figure 3.2: Two page embedding of $\Gamma_{IP}(D_8)$.

□

Theorem 3.3. *For a finite abelian group G, $bt(\Gamma_{IP}(G))$ is at most two if G is isomorphic to the trivial group of order one, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3$ or $\mathbb{Z}_4$.*

Proof. Since the intersection power graph of the trivial group of order one is a null graph, the book thickness for this graph is zero. Since the intersection power graph of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ and $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \cdots \times \mathbb{Z}_3$ have at least one edge, the book thickness for these graphs are at least one. Here each is a subgraph of the one page embeddable graph (see figure 3.3) for some integer n. Hence the book thickness for these graphs are one.

1 2 3 4 .... n a

Figure 3.3: One page embeddable graph.

Since the intersection power graph of $\mathbb{Z}_4$ is isomorphic to K_4. By Theorem 3.1, the book thickness for this graph is two □

4. Independence number of $\Gamma_{IP}(G)$

In the present section, $\alpha_0(\Gamma_{IP}(G))$ are attained.

Theorem 4.1. *For a finite group G with order $p_1^{\beta_1}p_2^{\beta_2}\cdots p_m^{\beta_m}$, where $p_1,p_2,\ldots,p_m$ are different primes and $\beta_1,\beta_2,\ldots,\beta_m$ are positive integers, $\alpha_0(\Gamma_{IP}(G)) \geq m$.*

Proof. Since each p_i is a divisor of $|G|$, G has elements a_i such that $o(a_i) = p_i$, for $1 \leq i \leq m$. Emphasize that $\langle a_i\rangle \cap \langle a_j\rangle = \{e\}$, for each $i \neq j$. From that $\{a_1,a_2,\ldots,a_m\}$ is an independent set of $\Gamma_{IP}(G)$. Hence the required result follows □

Theorem 4.2. *For a finite group with order $p_1^{\beta_1}p_2^{\beta_2}\cdots p_m^{\beta_m}$, where $p_1,p_2,\ldots,p_m$ are different primes and $\beta_1,\beta_2,\ldots,\beta_m$ are positive integers, $\alpha_0(\Gamma_{IP}(G)) = m \Leftrightarrow G$ has a unique subgroup with order p_i, for each $i = 1,2,\ldots,m$.*

Proof. Suppose that G has a unique subgroup with order p_i, for all $i = 1,2,\ldots,m$. For $\alpha_0(\Gamma_{IP}(G)) > m$, G has an independent set X with minimum $m+1$ elements. From Theorem 2.2, the orders of elements in X are relatively prime. Since $|G|$ has exactly m different prime divisors, we cannot find $m+1$ elements whose orders are relatively prime in G. From this $\alpha_0(\Gamma_{IP}(G)) \leq m$. Also, by Theorem 4.1, $\alpha_0(\Gamma_{IP}(G)) \geq m$. Hence $\alpha_0(\Gamma_{IP}(G)) = m$.

Conversely, suppose that $\alpha_0(\Gamma_{IP}(G)) = m$. By Cauchy's Theorem, G contain elements with order p_i, where $i = 1,2,\ldots,m$. Let $a_i \in G$ such that $o(a_i) = p_i$, where $i = 1,2,\ldots,m$. Suppose G has two different subgroups with order p_i, for some i. Let $b_i \in G$ such that $o(b_i) = p_i$. Obviously, $\langle a_i\rangle \cap \langle b_i\rangle = \{e\}$ and so $\{a_1,a_2,\ldots,a_m,b_i\}$ is an independent set in $\Gamma_{IP}(G)$ with $m+1$ elements, this is a contradiction. Therefore G has a unique subgroup with order p_i, where $i = 1,2,\ldots,m$ □

5. Conclusion

In this research article, we talked about certain properties of $\Gamma_{IP}(G)$. Also, we discussed about the book thickness and the independence number of $\Gamma_{IP}(G)$.

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