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Fuzzy Soft BCK-Modules

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Abstract. A BCK-module is an action of a BCK-algebra on an abelian group. In this paper, the theory of fuzzy soft sets is applied on BCK-modules and thereby introduced the notion of fuzzy soft BCK-module ($fsX-$module). In this regard, various algebraic operations between $fsX-$modules, like $fsX-$ intersection, $fsX-$union etc, are studied. Also, $fsX-$homormorphisms for $fsX-$modules is defined and discussed. Moreover, fuzzy soft exactness of $fsX-$modules is introduced and discussed.

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1. INTRODUCTION

Most of the problems that come across medical sciences, engineering, economics etc are uncertain in nature. In order to tackle these problems, Zadeh in [32], introduced the notion of fuzzy sets. The concept of fuzzy sets revolutionize the classical set theory in the sense of dealing with uncertainties. In coming years, some other tools like, intuitionistic fuzzy set [5], rough sets [26] were also developed to deal with these uncertainties. However, as in [25], these tools carried their own deficiencies. In 1999, Molodsov [25] carrying the spirit to deal with the models including uncertainties and vagueness, introduced the notion of soft sets. The theory of soft sets due to its richness attracted several applications. Some of these are given in [2, 10, 16, 20, 23, 30]. In 2001 [22], Maji et al. introduced the novel concept of fuzzy soft sets discussed its various attributes. The application of fuzzy soft sets in decision making problems were studied in [19, 21, 28]. The richness application of fuzzy related concepts, allowed its applications to more than just above said scope only. In this regard, fuzzy groups were studied by Rosenfield in [31], soft groups were introduced by Aktas and Cagman in [3]. Similarly, in [6], Aygunoglu and Aygun discussed fuzzy soft groups and Riaz et al, in [29], studied certain properties of bipolar fuzzy soft topology.

In the current work, Maji's concept of fuzzy soft sets, [22], is applied to the notion of BCK-modules and thereby introduced the notion fuzzy soft BCK-modules. The notion of BCK-modules was introduced in [1] as an action of a BCK-algebra X on an Abelian group M (see Definition 2.12). A BCK-algebra was introduced in 1966, Imai and Tanaka [13] as a natural generalization of BCK - propositional calculi [12] (see Definition 2.11). The notions of BCK-algebra and BCK-modules enjoyed attentions of several researchers over the years. Some of these can be seen in ([1, 7, 9, 11, 12, 13, 17, 27]). Recently, in [14], Jun applied the theory of soft sets to BCK/BCI-algebra and further in [15], Jun and Park explored the applications of soft set to the ideal theory of BCK/BCI-algebra. In [4], Alghamdi et al. studied the application of bipolar fuzzy sets on BCK-modules, further in [18], Kashif et al. introduced the notion of soft BCK-modules. Fuzzy soft BCK-modules are introduced in this paper as a generalization of soft BCK-modules.

2. Preliminaries

In this section, some preliminaries from the theories of soft sets, fuzzy soft sets, BCK-modules are included. Throughout this section, following notational convention is adopted: $\mathfrak{U}$ is an initial universe, $P(\mathfrak{U})$ is the power set of $\mathfrak{U}$, E is a parameter set, A, B, C are its subsets, $\mathfrak{I}^X$ denotes set of all fuzzy sets on X.

Definition 2.1. (Soft Set) [25]
A pair $(\mathfrak{F}, A)$ is called soft set over universe $\mathfrak{U}$, where $\mathfrak{F}$ is a mapping given by $\mathfrak{F} : A \to P(\mathfrak{U})$.

An example of soft set related to packing of fruits is furnished below:

Example 2.2. *Let $\mathfrak{U} = \{r(red), g(green), b(blue)\}$ be the universe consisting of different colour boxes and E be the set of parameters consisting of different fruits. If $A = \{$Apple, Mango, Orange, Peach$\} \subset E$, then a soft set $(\mathfrak{F}, A)$ related to the packing of fruits in different colour boxes is given by:*

$$\mathfrak{F}(p) = \begin{cases} \{r\}, & \text{if } p \text{ is Apple} \\ \{g, b\}, & \text{if } p \text{ is Peach} \\ \{r, g, b\}, & \text{if } p \text{ is Orange} \\ \{b\}, & \text{if } p \text{ is Mango} \end{cases}$$

The details of soft union, soft intersection, soft inclusion and other related terms can be seen in [23].

Definition 2.3. (Fuzzy Soft Set) [22]
Let $\mathfrak{I}^X$ denotes the collection of all fuzzy sets over X and A is a subset of the set of

parameters E. If $\mathfrak{f} : A \to \Im^X$ is a mapping, that is, for all $a \in A$, $\mathfrak{f}(a) = \mathfrak{f}_a$, belongs to $\Im^X$, then the pair $(\mathfrak{f}, A)$ is called a fuzzy soft set (fss) over X.

The following example is presented here to elaborate the notion of fss:

Example 2.4. *Consider an example of food quality test by the administration of a fast food restaurant. Let $U = \{u_1(\text{sandwich}), u_2(\text{pizza}), u_3(\text{burger})\}$ be the set of fast foods at the restaurant and E be its customers on a certain day of the week. Suppose $A_1 = \{a_1, a_2, a_3\}$, and $A_2 = \{a_2, a_3, a_4\}$ be two families participating in the food quality test. They are required to eat the food and then share their taste experiences in a range of 0 to 1. The fss $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ over U, representing the ratting of fast food quality is given by following tables;*

	$(\mathfrak{f}_1, A_1)$			$(\mathfrak{f}_2, A_2)$		
	a_1	a_2	a_3	a_2	a_3	a_4
u_1	0.1	0	0.3	0	0.1	0.3
u_2	0.5	0.2	0.5	0.3	0.5	0.5
u_3	0.4	0.7	0.1	0.6	0.4	0.1

Definition 2.5. (Fuzzy Soft Inclusion)[22]
Let $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ be two fss over X. Then $(\mathfrak{f}_1, A_1)$ is a fuzzy soft subset of $(\mathfrak{f}_2, A_2)$ if

(1) $A_1 \subseteq A_2$ *and*
(2) $\forall a \in A_1$, $\mathfrak{f}_{1a}$ *is a fuzzy subset of* $\mathfrak{f}_{2a}$

The fuzzy soft inclusion is denoted by $(\mathfrak{f}_1, A_1) \subseteq (\mathfrak{f}_2, A_2)$.

Definition 2.6. (Fuzzy Soft Union) [22]
Let $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ be two fss over X. Then their $fs-$union is the fss $(\mathfrak{h}, B)$ over X, where $B = A_1 \cup A_2$ and $\forall\ b \in B$, defined by

$$\mathfrak{h}(b) = \begin{cases} \mathfrak{f}_{1b}, & \text{if } b \in A_1 - A_2. \\ \mathfrak{f}_{2b}, & \text{if } b \in A_2 - A_1. \\ \mathfrak{f}_{1b} \vee \mathfrak{f}_{2b}, & \text{if } b \in A_1 \cap A_2. \end{cases}$$

The above relationship is denoted by $(\mathfrak{f}_1, A_1) \cup (\mathfrak{f}_2, A_2) = (\mathfrak{h}, B)$.

Definition 2.7. (Fuzzy Soft Intersection) [22]
Let $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ be two fss over X. Then their $fs-$intersection is the fss $(\mathfrak{g}, C)$ over X, where $C = A_1 \cap A_2$ and

$$\mathfrak{g}_c = \mathfrak{f}_{1c} \wedge \mathfrak{f}_{2c}, \forall\ c \in C.$$

It is denoted by $(\mathfrak{f}_1, A_1) \cap (\mathfrak{f}_2, A_2) = (\mathfrak{g}, C)$.

The following example is produced to elaborate above operations on fss.

Example 2.8. *Continuing the Example 2.4, the $fs-$union $(\mathfrak{h}, B)$ and $fs-$intersection $(\mathfrak{g}, C)$ of the fss $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ is given by;*

	$(\mathfrak{h},$	$B)$			$(\mathfrak{g},$	$C)$
	a_1	a_2	a_3	a_4	a_2	a_3
u_1	0.1	0	0.3	0.3	0	0.1
u_2	0.5	0.3	0.5	0.5	0.2	0.5
u_3	0.4	0.7	0.4	0.1	0.6	0.1

Definition 2.9. (Fuzzy Soft AND operation) [22]
*Let $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ be two fss over X. Then $(\mathfrak{f}_1, A_1)$**AND**$(\mathfrak{f}_2, A_2)$ is an fss over X, denoted by $(\mathfrak{f}_1, A_1) \wedge (\mathfrak{f}_2, A_2)$ and it is defined as $(h, A_1 \times A_2)$ and $h(a_1, a_2) = h_{a_1,a_2} = f_{1a_1} \wedge f_{2a_2}, \forall (a_1, a_2) \in A_1 \times A_2$.*

Example 2.10. *Consider the fss $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ over U defined in the Example 2.4. Then $(a_1, a_2), (a_1, a_3), (a_1, a_4), (a_2, a_2), (a_2, a_3), (a_2, a_4), (a_3, a_2), (a_3, a_3), (a_3, a_4)$ are all the elements of the cartesian product $A_1 \times A_2$. The fss $(h, A_1 \times A_2)$ over U, where $h = \mathfrak{f}_1 \wedge \mathfrak{f}_2$; is given in following table:*

					(h, B)				
	(a_1, a_2)	(a_1, a_3)	(a_1, a_4)	(a_2, a_2)	(a_2, a_3)	(a_2, a_4)	(a_3, a_2)	(a_3, a_3)	(a_3, a_4)
u_1	0	0.1	0.1	0	0	0	0	0.1	0.3
u_2	0.3	0.5	0.5	0.2	0.2	0.2	0.3	0.5	0.5
u_3	0.4	0.4	0.1	0.6	0.4	0.1	0.1	0.1	0.1

Next, we include some basics of BCK-algebras and BCK-modules.

Definition 2.11. (BCK-Algebra) [13]
A BCK-algebra is an algebraic system $(X, *, 0)$ that satisfy the following axioms for all $a, b, c \in X$: **BCK-1**) $((a * b) * (a * c)) * (c * b) = 0$; **BCK-2**) $(a * (a * b)) * b = 0$; **BCK-3**) $a * a = 0$; **BCK-4**) $0 * a = 0$; **BCK-5**) $a * b = 0, b * a = 0$ *implies* $a = b$; **BCK-6**) $a * b = 0$ *iff* $a \leq b$.

Definition 2.12. (BCK-Module) [1]
Let $(X, *, 0)$ be a BCK-algebra and $(M, +)$ be an abelian group. Then M is said to be a (left)BCK-module if $(x, m) \mapsto xm$ is a mapping from $X \times M \to M$, such that:

BCKM-1: $(x \wedge y)m = x(ym)$
BCKM-2: $x(m_1 + m_2) = xm_1 + xm_2$
BCKM-3: $0m = 0$, *for all* $x, y \in X$ *and* $m_1, m_2 \in M$.
BCKM-4: $1m = m$, *whenever,* X *is bounded.*

where, $(x \wedge y) = y * (y * x)$.

In sequel, a BCK-algebra $(X, *, 0)$ will be denoted by X. If X contain an element 1, such that $a \leq 1 \quad \forall a \in X$, then X is called bounded. X is said to be commutative if $a \wedge b = b \wedge a$ holds for all $a, b \in X$, where $a \wedge b = b * (b * a)$. A BCK-algebra X is said to be implicative if $a * (b * a) = a$ for all $a, b \in X$. For undefined notions and details of BCK-algebras and BCK-modules, the reader is referred to [1, 7, 13, 17, 27].

In [1, 7, 18, 17], several examples of $X-$modules were given. Here, we present another example of an $X-$module.

Example 2.13. *Consider the BCK-algebra $(X, *, 0)$ as in Table 1. It can be verified easily that X is bounded, commutative and non-implicative. For a subset $M_1 = \{0, 1\}$ of X, define an operation addition $+$ on M_1 by Table 2. Then $(M_1, +)$ is indeed an abelian group. Further, define the operation of scalar multiplication " $\cdot$ " from $X \times M_1 \mapsto M_1$ by $(x, m) = x \cdot m = x \wedge m$ in Table 3. It can be verified easily from Table 2-3 that M_1 satisfies all the axioms of Definition 2.12. Hence, M_1 is an $X-$module. Also, with similar scalar multiplication, it can be seen that the abelian group $(M_2, +)$ given in Table 4, where $M_2 = \{0, 2\}$, forms an $X-$module.*

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	0
2	2	1	0	0	0	0
3	3	2	1	0	0	0
4	4	3	2	1	0	0
5	5	4	3	2	1	0

TABLE 1

$\wedge$	0	1
0	0	0
1	0	1
2	0	1
3	0	1
4	0	1
5	0	1

TABLE 2

+	0	1
0	0	1
1	1	0

TABLE 3

+	0	2
0	0	2
2	2	0

TABLE 4

3. FUZZY SOFT BCK MODULES

In [8], Bakhshi studied the applications of fuzzy set theory on BCK-modules. Alghamdi et al in [4] introduced the notion of bipolar fuzzy BCK-modules and discussed its various attributes. The soft sets theory was applied on BCK modules by Kashif et al in [18]. They introduced the notion of soft BCK-modules and explored some algebraic attributes. Three isomorphism theorems for soft BCK-modules were established in this regard. In this section, the notion of fuzzy soft BCK-modules is introduced and discussed for its algebraic characterizations. All through this section, M represents an $X-$module.

Definition 3.1. (Fuzzy Soft BCK-Module)
Let $(\mathfrak{f}, A)$ be an fss over an $X-$module M. Then we call $(\mathfrak{f}, A)$ a fuzzy soft BCK-module ($fsX-$module) over M, if $\forall a \in A$, $m_1, m_2 \in M$ and $x \in X$, following are satisfied:

FSXM-1: $\mathfrak{f}_a(m_1 + m_2) \geq \min\{\mathfrak{f}_a(m_1), \mathfrak{f}_a(m_2)\}$
FSXM-2: $\mathfrak{f}_a(-m) = \mathfrak{f}_a(m)$
FSXM-3: $\mathfrak{f}_a(0) = 1$
FSXM-4: $\mathfrak{f}_a(xm) \geq \mathfrak{f}_a(m)$

The collection of all $fsX-$module over an $X-$module M is denoted by $fsX(M)$.

The following example illustrates the above definition.

Example 3.2. *Consider the $X-$module M_1 discussed in Example 2.13. Let $A_1 = \{a, b\}$ and $A_2 = \{b, c\}$ be any two parametric sets. Then the fss $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ over $X-$module M_1 are given in given in Table 5 and Table 6. It can seen from Tables 2-6 that $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ are $fsX-$modules over the $X-$module M_1.*

	0	*1*
$\mathfrak{f}_{1a}$	*1*	*0.8*
$\mathfrak{f}_{1b}$	*1*	*0.7*

TABLE 5

	0	*1*
$\mathfrak{f}_{2b}$	*1*	*0.5*
$\mathfrak{f}_{2c}$	*1*	*0.4*

TABLE 6

Proposition 3.3. *Let $(\mathfrak{f}, A) \in fsX(M)$. Then $\mathfrak{f}_a(0) \geq f_a(m)$ for all $a \in A$ and $m \in M$.*

Proof. Easy to proof. □

Theorem 3.4. *Let $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ be two $fsX-$modules over M. Then $(\mathfrak{f}_1, A_1) \cap (\mathfrak{f}_2, A_2) \in fsX(M)$.*

Proof. The $fs-$intersection $(\mathfrak{f}_1, A_1) \cap (\mathfrak{f}_2, A_2) = (\mathfrak{g}, C)$ is an fss (cf. 2.7). It requires to show that $(\mathfrak{g}, C)$ satisfies axioms of $fsX-$module (cf. 3.1). Let $c \in C = A_1 \cap A_2$ and $m_1, m_2 \in M$. Then following are implications of Definition 2.7:

$$\begin{aligned}
\mathfrak{g}_c(m_1 + m_2) &= (\mathfrak{f}_{1c} \wedge \mathfrak{f}_{2c})(m_1 + m_2) \\
&= \min\{\mathfrak{f}_{1c}(m_1 + m_2), \mathfrak{f}_{2c}(m_1 + m_2)\} \\
&\geq \min\{\min\{\mathfrak{f}_{1c}(m_1), \mathfrak{f}_{1c}(m_2)\}, \min\{\mathfrak{f}_{2c}(m_1), \mathfrak{f}_{2c}(m_2)\}\} \\
&= \min\{\min\{\mathfrak{f}_{1c}(m_1), \mathfrak{f}_{2c}(m_1)\}, \min\{\mathfrak{f}_{1c}(m_2), \mathfrak{f}_{2c}(m_2)\}\} \\
&\geq \min\{(\mathfrak{f}_{1c} \wedge \mathfrak{f}_{2c})(m_1), (\mathfrak{f}_{1c} \wedge \mathfrak{f}_{2c})(m_2)\} \\
&= \min\{\mathfrak{g}_c(m_1), \mathfrak{g}_c(m_2)\}
\end{aligned}$$

Similarly, for $x \in X$ and $m_1 \in M$, Definition 2.7 implies the following:

$$\begin{aligned}
\mathfrak{g}_c(xm_1) &= (\mathfrak{f}_{1c} \wedge \mathfrak{f}_{2c})(xm_1) \\
&= \min\{\mathfrak{f}_{1c}(xm_1), \mathfrak{f}_{2c}(xm_1)\} \\
&\geq \min\{\mathfrak{f}_{1c}(m_1), \mathfrak{f}_{2c}(m_1)\} \\
&= (\mathfrak{f}_{1c} \wedge \mathfrak{f}_{2c})(m_1) = \mathfrak{g}_c(m_1)
\end{aligned}$$

The remaining axioms of Definition 3.1 are easy to show. □

Theorem 3.5. *Let $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ be two $fsX-$modules over M. Then $(\mathfrak{f}_1, A_1) \cup (\mathfrak{f}_2, A_2) \in fsX(M)$, whenever $A_1 \cap A_2 = \phi$.*

Proof. The $fs-$union $(\mathfrak{f}_1, A_1) \cup (\mathfrak{f}_2, A_2) = (\mathfrak{h}, B)$ is a fss (cf. 2.6). It remains to show that $(\mathfrak{h}, B)$ satisfies all the axioms of $fsX-$module (cf. 3.1). Since $A_1 \cap A_2 = \phi$, therefore, $\forall b \in B$ either $b \in A_1$ or $b \in A_2$. Therefore, Definition 2.6 implies that either $\mathfrak{h} = \mathfrak{f}_1$ or $\mathfrak{h} = \mathfrak{f}_2$ which are both $fsX-$modules. This completes the proof. □

Theorem 3.6. *Let $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ be two $fsX-$modules over M. Then $(\mathfrak{f}_1, A_1) \wedge (\mathfrak{f}_2, A_2) \in fsX(M)$.*

Proof. It is clear from Definition 2.9 that $(h, A_1 \times A_2) = (\mathfrak{f}_1, A_1) \wedge (\mathfrak{f}_2, A_2)$ is a fss over M. The only thing remains to show is that $(h, A_1 \times A_2)$ satisfies Definition 3.1. Let $(a_1, a_2) \in A_1 \times A_2$ and $m_1, m_2 \in M$. Then following are implications of Definition 2.7:

$$\begin{aligned} h_{a_1,a_2}(m_1 + m_2) &= (\mathfrak{f}_{1a_1} \wedge \mathfrak{f}_{2a_2})(m_1 + m_2) \\ &= \min\{\mathfrak{f}_{1a_1}(m_1 + m_2), \mathfrak{f}_{2a_2}(m_1 + m_2)\} \\ &\geq \min\{\min\{\mathfrak{f}_{1a_1}(m_1), \mathfrak{f}_{1a_1}(m_2)\}, \min\{\mathfrak{f}_{2a_2}(m_1), \mathfrak{f}_{2a_2}(m_2)\}\} \\ &= \min\{\min\{\mathfrak{f}_{1a_1}(m_1), \mathfrak{f}_{2a_2}(m_1)\}, \min\{\mathfrak{f}_{1a_1}(m_2), \mathfrak{f}_{2a_2}(m_2)\}\} \\ &\geq \min\{(\mathfrak{f}_{1a_1} \wedge \mathfrak{f}_{2a_2})(m_1), (\mathfrak{f}_{1a_1} \wedge \mathfrak{f}_{2a_2})(m_2)\} \\ &= \min\{h_{a_1,a_2}(m_1), h_{a_1,a_2}(m_2)\} \end{aligned}$$

Similarly, for $x \in X$ and $m_1 \in M$, Definition 2.7 implies the following:

$$\begin{aligned} h_{a_1,a_2}(xm_1) &= (\mathfrak{f}_{1a_1} \wedge \mathfrak{f}_{2a_2})(xm_1) \\ &= \min\{\mathfrak{f}_{1a_1}(xm_1), \mathfrak{f}_{2a_2}(xm_1)\} \\ &\geq \min\{(\mathfrak{f}_{1a_1}(m_1), \mathfrak{f}_{2a_2}(m_1)\} \\ &= (\mathfrak{f}_{1a_1} \wedge \mathfrak{f}_{2a_2})(m_1) = h_{a_1,a_2}(m_1) \end{aligned}$$

The remaining axioms of Definition 3.1 can be shown in similar way. □

Example 3.7. *Consider the $X-$modules $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ discussed in Example 3.2. Here, $B = A_1 \cup A_2 = \{a, b, c\}$, $C = A_1 \cap A_2 = \{b\}$, and $A_1 \times A_2 = \{(a, b), (a, c), (b, b), (b, c), \}$. Therefore the $fs-$union $(\mathfrak{h}, B)$, $fs-$intersection $(\mathfrak{g}, C)$ and $fs-$and operation $(h, A_1 \times A_2)$ of $fsX-$modules $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ is given in following Tables:*

	$(\mathfrak{h}, B)$			$(\mathfrak{g}, C)$	$(h,$	$A_1\times$	$A_2)$	
	a	b	c	b	(a, b)	(a, c)	(b, b)	(b, c)
0	1	1	1	1	1	1	1	1
1	0.8	0.7	0.4	1	1	0.4	0.5	0.4

Next, we present the notion of $fs-$function between two fss.

Definition 3.8. (Fuzzy Soft Function on fss)
Let $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ be two fss over U and V, correspondingly. If $\alpha : U \to V$ and $\beta : A_1 \to A_2$ be two functions, then the pair (α, β) is called a fuzzy soft function ($fs-$function) from the fss $(\mathfrak{f}_1, A_1)$ to $(\mathfrak{f}_2, A_2)$.

The Image and preimage of the $fs-$funcition $(\alpha, \beta) : (\mathfrak{f}_1, A_1) \to (\mathfrak{f}_1, A_1)$ are given below:

(1) *The image of $(\mathfrak{f}_1, A_1)$ under the $fs-$function (α, β), represented as $(\alpha, \beta)(\mathfrak{f}_1, A_1)$, is the fss over V defined by $(\alpha, \beta)(\mathfrak{f}_1, A_1) = (\alpha(\mathfrak{f}_1), \beta(A_1))$, where*

$$\alpha(\mathfrak{f}_1)_k(q) = \begin{cases} \bigvee_{\alpha(p)=q} \bigvee_{\beta(a)=k} \mathfrak{f}_{1a}(p) & \text{if } p \in \alpha^{-1}(q) \\ 0 & \text{elseways} \end{cases}, \quad \forall\, k \in \beta(A_1), q \in V.$$

(2) *For the $fs-$function (α, β), the pre-image of $(\mathfrak{f}_2, A_2)$, represented by $(\alpha, \beta)^{-1}$ $(\mathfrak{f}_2, A_2)$, is the fss over U, defined by $(\alpha, \beta)^{-1}(\mathfrak{f}_2, A_2) = (\alpha^{-1}(\mathfrak{f}_2), \beta^{-1}(A_2))$,*

where

$$\alpha^{-1}(\mathfrak{f}_2)_a(p) = \mathfrak{f}_{2\beta(a)}(\alpha(p)), \quad \forall\, a \in \beta^{-1}(A_2),\ p \in U.$$

If α and β is $1-1$(onto), then (α, β) is said to be $1-1$(onto).

Example 3.9. *Let $A_1 = \{a_1, a_2\}$ and $A_2\{a_3, a_4\}$ be subset of the parametric set E and $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ be two fss over $U = \{u_1, u_2\}$ and $V = \{v_1, v_2\}$ respectively given in Tables 7 and 8. Define the functions $\alpha : U \to V$ and $\beta : A_1 \to A_2$ as follows: $\alpha(u_1) = v_2, \alpha(u_2) = v_1$ and $\beta(a_1) = a_3, \beta(a_2) = a_4$. Then the image $(\alpha, \beta)(\mathfrak{f}_1, A_1)$ and the preimage $(\alpha, \beta)^{-1}(\mathfrak{f}_2, A_2)$ of the fss $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ respectively under the $fs-$function (α, β) is given in the Tables 9 and 10 respectively.*

U	$(\mathfrak{f}_1, A_1)$ a_1	a_2
u_1	0.1	0
u_2	0.5	0.2

TABLE 7

V	$(\mathfrak{f}_2, A_2)$ a_3	a_4
v_1	0	0.1
v_1	0.3	0.5

TABLE 8

V	(α, β) a_3	$(\mathfrak{f}_1, A_1)$ a_4
v_1	0.5	0.2
v_2	0.1	0

TABLE 9

U	$(\alpha, \beta)^{-1}$ a_1	$(\mathfrak{f}_2, A_2)$ a_2
u_1	0.3	0.5
u_2	0	0.1

TABLE 10

Definition 3.10. ($fsX-$Module Homomorphism)
Let $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ be two $fsX-$modules over $X-$modules M_1 and M_2, respectively. If α is an $X-$homomorphism from $M_1 \to M_2$(cf. [1]) and $\beta : A_1 \to A_2$ is a mapping between the parameters set, then the $fs-$function $(\alpha, \beta) : (\mathfrak{f}_1, A_1) \to (\mathfrak{f}_1, A_1)$ is called an $fsX-$module homomorphism ($fsX-$homomorphism).
If α is an $X-$isomorphism from $M_1 \to M_2$ and β is $1-1$ mapping from A_1 onto A_2, then (α, β) is said to be a $fsX-$module isomorphism ($fsX-$isomorphism).

(1) *The image of the $fsX-$homomorphism $(\alpha, \beta) : (\mathfrak{f}_1, A_1) \to (\mathfrak{f}_2, A_2)$ is defined as*

$$\mathrm{Im}(\alpha, \beta) = \{\alpha(\mathfrak{f}_1)_k(n)\}$$

(2) *The kernel of the $fsX-$homomorphism $(\alpha, \beta) : (\mathfrak{f}_1, A_1) \to (\mathfrak{f}_2, A_2)$ is defined as*

$$\mathrm{Ker}(\alpha, \beta) = \{\mathfrak{f}_{1a}(m) \,|\, \alpha(\mathfrak{f}_1)_k(n) = 1\}, \quad \forall\, k \in \beta(A_1)\,\&\, \alpha(m) = n \in M_2.$$

Example 3.11. *Continuing Example 3.2, let $M_1 = \{0, 1\} = M_2$ be $X-$modules and define the $X-$homomorphism $\alpha : M_1 \to M_2$ and the mapping $\beta : A_1 \to A_2$ as follows: $\alpha(0) = 0, \alpha(1) = 1$ and $\beta(a) = c, \beta(b) = b$. Then the image $(\alpha, \beta)(\mathfrak{f}_1, A_1)$ and the $\mathrm{Ker}(\alpha, \beta)$ of the $fsX-$homomorphism (α, β) is given in the Tables 11 and 12 respectively.*

M_2	(α,β) b	$(\mathfrak{f}_1, A_1)$ c
0	1	1
1	0.7	0.8

TABLE 11

U	Ker a	(α,β) b
0	1	1
1	0	0

TABLE 12

Theorem 3.12. *Let $(\mathfrak{f}_1, A_1) \in fsX(M_1)$ and $(\mathfrak{f}_2, A_2) \in fsX(M_2)$. If (α,β) is a monic $fsX-$homomorphism from $(\mathfrak{f}_1, A_1)$ to $(\mathfrak{f}_2, A_2)$, then the image of $(\mathfrak{f}_1, A_1)$ is an $fsX-$module over M_2.*

Proof. Let $k \in \beta(A_1)$ and $n_1, n_2 \in M_2$. If $\alpha^{-1}(n_1) = \emptyset$ or $\alpha^{-1}(n_2) = \emptyset$; the proof is straightforward. Let assume that there exist $m_1, m_2 \in M_1$ such that $\alpha(m_1) = n_1$, $\alpha(m_2) = n_2$.

$$\begin{aligned}
\alpha(\mathfrak{f}_1)_k(n_1+n_2) &= \bigvee_{\alpha(t)=n_1+n_2} \bigvee_{\beta(a)=k} \mathfrak{f}_{1a}(t) \quad (cf.3.8)\\
&\geq \bigvee_{\beta(a)=k} \mathfrak{f}_{1a}(m_1+m_2)\\
&\geq \bigvee_{\beta(a)=k} \min\{\mathfrak{f}_{1a}(m_1), \mathfrak{f}_{1a}(m_2)\} \quad (cf.3.1)\\
&\geq \min\{\bigvee_{\beta(a)=k} \mathfrak{f}_{1a}(m_1), \bigvee_{\beta(a)=k} \mathfrak{f}_{1a}(m_2)\}.
\end{aligned}$$

This above inequality is satisfied for each $m_1, m_2 \in M_1$, such that $\alpha(m_1) = n_1$, $\alpha(m_2) = n_2$. This implies

$$\begin{aligned}
\alpha(\mathfrak{f}_1)_k(n_1+n_2) &\geq \min\{\bigvee_{\alpha(t_1)=n_1} \bigvee_{\beta(a)=k} \mathfrak{f}_{1a}(t_1), \bigvee_{\alpha(t_2)=n_2} \bigvee_{\beta(a)=k} \mathfrak{f}_{1a}(t_2)\}\\
&= \min\{\alpha(\mathfrak{f}_1)_k(n_1), \alpha(\mathfrak{f}_1)_k(n_2)\} \quad (cf.3.8).
\end{aligned}$$

The remaining axioms of Definition 3.1 are easy to see. This completes the proof. □

Theorem 3.13. *Let $(\mathfrak{f}_1, A_1) \in fsX(M_1)$ and $(\mathfrak{f}_2, A_2) \in fsX(M_2)$. If (α,β) is a monic $fsX-$homomorphism from $(\mathfrak{f}_1, A_1)$ to $(\mathfrak{f}_2, A_2)$, then the preimage of $(\mathfrak{f}_2, A_2)$, is an $fsX-$module over M_1.*

Proof. Let $a \in \beta^{-1}(A_2)$ & $m_1, m_2 \in M_1$. Then following are implications from Definition 3.1:

$$\begin{aligned}\alpha^{-1}(\mathfrak{f}_2)_a(m_1+m_2) &= \mathfrak{f}_{2\beta(a)}(\alpha(m_1+m_2))\\ &= \mathfrak{f}_{2\beta(a)}(\alpha(m_1)+\alpha(m_2))\\ &\geq \min\{\mathfrak{f}_{2\beta(a)}\alpha(m_1), \mathfrak{f}_{2\beta(a)}\alpha(m_2)\}\\ &= \min\{\alpha^{-1}(\mathfrak{f}_2)_a(m_1), \alpha^{-1}(\mathfrak{f}_2)_a(m_2)\}.\end{aligned}$$

The remaining conditions of Definition 3.1 can be presented in a similar manner. This completes the proof. □

4. $fs-$EXACTNESS OF $X-$MODULES

In section 5 from [18], the authors introduced the notion of soft $X-$exactness of BCK-modules and discussed its various attributes. In the current section, the notion of fuzzy soft exactness is introduced as a generalization of soft $X-$exactness. All the sets considered in this final section, are $X-$modules. The notion of $fs-$exact sequence of $X-$modules is introduced and discussed different algebraic attributes.

Definition 4.1. (Fuzzy Soft Exactness)
Let $(\mathfrak{f}_1, A_1)$,$(\mathfrak{f}_2, A_2)$ and $(\mathfrak{f}_3, A_3)$ be three $fsX-$modules over $X-$modules M_1, M_2 and M_3, correspondingly and (α_1, β_1) and (α_2, β_2) be two $fsX-$ homomorphisms from $(\mathfrak{f}_1, A_1)$ to $(\mathfrak{f}_2, A_2)$ and $(\mathfrak{f}_2, A_2)$ to $(\mathfrak{f}_3, A_3)$ respectively. Then the sequence

$$(\mathfrak{f}_1, A_1) \xrightarrow{(\alpha_1,\beta_1)} (\mathfrak{f}_2, A_2) \xrightarrow{(\alpha_2,\beta_2)} (\mathfrak{f}_3, A_3)$$

is called fuzzy soft exact (fse) at $(\mathfrak{f}_2, A_2)$, if $\mathrm{Im}(\alpha_1, \beta_1) = \mathrm{Ker}(\alpha_2, \beta_2)$,
An arbitrary sequence of $fsX-$modules

$$\cdots \to (\mathfrak{f}_{i-1}, A_{i-1}) \xrightarrow{(\alpha_{i-1},\beta_{i-1})} (\mathfrak{f}_i, A_i) \xrightarrow{(\alpha_i,\beta_i)} (\mathfrak{f}_{i+1}, A_{i+1}) \to \cdots$$

is fse if every $(\mathfrak{f}_i, A_i)$, $i \in I$ is fse.

Definition 4.2. *If $M_1 = 0$, then the fss $(\mathfrak{f}_1, A_1)$ on M_1, denoted by $(\mathfrak{f}_1, A_1) = 1$, is an $fsX-$module.*

Proposition 4.3. *Let $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ be two $fsX-$modules over $X-$modules M_1 and M_2, correspondingly. If (α_0, β_0) and (α_1, β_1) are two $fsX-$homomorphisms from 0 to M_1 and M_1 to M_2, correspondingly. Then (α_1, β_1) is monic iff $1 \xrightarrow{(\alpha_0,\beta_0)} (\mathfrak{f}_1, A_1) \xrightarrow{(\alpha_1,\beta_1)} (\mathfrak{f}_2, A_2)$ is fse.*

Proof. Since $0 \xrightarrow{\alpha_0} M_1 \xrightarrow{\alpha_1} M_2$ is $X-$exact, therefore from Definition 3.10, $\mathrm{Im}(\alpha_0, \beta_0) = 1$. Also, $\mathrm{Ker}(\alpha_1, \beta_1) = \{\mathfrak{f}_{1a}(p) \mid \alpha(\mathfrak{f}_1)_k(q) = 1\}$ and $q = 0$. Since (α_1, β_1) is monic, so $p = 0$ and $\mathrm{Ker}(\alpha_1, \beta_1) = 1$. This implies $\mathrm{Im}(\alpha_0, \beta_0) = \mathrm{Ker}(\alpha_1, \beta_1)$ and therefore,
$1 \xrightarrow{(\alpha_0,\beta_0)} (\mathfrak{f}_1, A_1) \xrightarrow{(\alpha_1,\beta_1)} (\mathfrak{f}_2, A_2)$ is fse.
Conversely, if $1 \xrightarrow{(\alpha_0,\beta_0)} (\mathfrak{f}_1, A_1) \xrightarrow{(\alpha_1,\beta_1)} (\mathfrak{f}_2, A_2)$ is fse, then $\mathrm{Im}(\alpha_0, \beta_0) = \mathrm{Ker}(\alpha_1, \beta_1) = 1$. This implies (α_1, β_1) is monic. This completes the proof. □

Proposition 4.4. *Let $(\mathfrak{f}_1, A_1)$ and $(\mathfrak{f}_2, A_2)$ be two $fsX-$modules over $X-$modules M_1 and M_2, respectively.*

(1) *If* $(\alpha_1, \beta_1) : (\mathfrak{f}_1, A_1) \to (\mathfrak{f}_2, A_2)$ *is epic, then* $\mathrm{Im}(\alpha_1, \beta_1) = (\mathfrak{f}_2, A_2)$.
(2) *If* $(\mathfrak{f}_1, A_1) \xrightarrow{(\alpha_1,\beta_1)} 1$*, then* $\mathrm{Ker}(\alpha_1, \beta_1) = (\mathfrak{f}_1, A_1)$ *and* (α_1, β_1) *is epic.*
(3) *If* $1 \xrightarrow{(\alpha_1,\beta_1)} (\mathfrak{f}_1, A_1)$*, then* $\mathrm{Im}(\alpha_1, \beta_1) = (\mathfrak{f}_1, A_1)$ *and* (α_1, β_1) *is monic.*
(4) *If* $(\alpha_1, \beta_1) : (\mathfrak{f}_1, A_1) \to (\mathfrak{f}_2, A_2)$ *is monic, then* $\mathrm{Ker}(\alpha_1, \beta_1) = 1$.
(5) *If* $1 \to (\mathfrak{f}_1, A_1) \xrightarrow{(\alpha_1,\beta_1)} (\mathfrak{f}_2, A_2) \to 1$ *is* fse*, then* (α_1, β_1) *is an* $fsX-$*isomorphism.*

Proof. The proofs are easy to see. □

Proposition 4.5. *Let* $(\mathfrak{f}_1, A_1)$ *and* $(\mathfrak{f}_2, A_2)$ *be two* $fsX-$*modules over* $X-$*modules* M_1 *and* M_2*, respectively. If* (α_1, β_1) *and* (α_2, β_2) *are two* $fsX-$*homomorphisms from* M_1 *to* M_2 *and* M_2 *to* 0*, respectively. Then* (α_1, β_1) *is epic iff* $(\mathfrak{f}_1, A_1) \xrightarrow{(\alpha_1,\beta_1)} (\mathfrak{f}_2, A_2) \xrightarrow{(\alpha_2,\beta_2)} 1$ *is* fse*.*

Proof. Let $(\mathfrak{f}_1, A_1) \xrightarrow{(\alpha_1,\beta_1)} (\mathfrak{f}_2, A_2) \xrightarrow{(\alpha_2,\beta_2)} 1$ be fse. Then $\mathrm{Im}(\alpha_1, \beta_1) = \mathrm{Ker}(\alpha_2, \beta_2)$. Since $\mathrm{Ker}(\alpha_2, \beta_2) = (\mathfrak{f}_2, A_2)$, (c.f. 4.4). Therefore, $\mathrm{Im}(\alpha_1, \beta_1) = (\mathfrak{f}_2, A_2)$, and hence (α_1, β_1) is epic.
Conversely, let (α_1, β_1) be epic. This implies $\mathrm{Ker}(\alpha_2, \beta_2) = (\mathfrak{f}_2, A_2) = \mathrm{Im}(\alpha_1, \beta_1)$, (c.f. 4.4). Hence, $\mathrm{Im}(\alpha_1, \beta_1) = \mathrm{Ker}(\alpha_2, \beta_2)$. This completes the proof. □

Lemma 4.6. *Let* $(\mathfrak{f}_1, A_1), (\mathfrak{f}_2, A_2)$ *and* $(\mathfrak{f}_3, A_3)$ *be three* $fsX-$*modules over* $X-$*modules* M_1, M_2, M_3 *respectively. If* $(\mathfrak{f}_1, A_1) \xrightarrow{(\alpha_1,\beta_1)} (\mathfrak{f}_2, A_2) \xrightarrow{(\alpha_2,\beta_2)} (\mathfrak{f}_2, A_2)$ *is a* fse *with* (α_1, β_1) *epic and* (α_2, β_2) *monic, then* $(\mathfrak{f}_2, A_2) = 1$.

Proof. Since $(\mathfrak{f}_1, A_1) \xrightarrow{(\alpha_1,\beta_1)} (\mathfrak{f}_2, A_2) \xrightarrow{(\alpha_2,\beta_2)} (\mathfrak{f}_3, A_3)$ is fse and (α_1, β_1) is epic, therefore from Proposition 4.4, $\mathrm{Im}(\alpha_1, \beta_1) = \mathrm{Ker}(\alpha_2, \beta_2)$ and $\mathrm{Im}(\alpha_1, \beta_1) = (\mathfrak{f}_2, A_2)$. This implies $\mathrm{Ker}(\alpha_2, \beta_2) = (\mathfrak{f}_2, A_2)$. Also, since (α_2, β_2) is monic, therefore, from Proposition 4.4, $\mathrm{Ker}(\alpha_2, \beta_2) = 1$. Hence, $(\mathfrak{f}_2, A_2) = 1$. This completes the proof. □

Lemma 4.7. *Let* $(\mathfrak{f}_i, A_i)$ *be* $fsX-$*modules over the* $X-$*modules* M_i $i = 1, 2, 3, 4$*, respectively. If* $(\mathfrak{f}_1, A_1) \xrightarrow{(\alpha_1,\beta_1)} (\mathfrak{f}_2, A_2) \xrightarrow{(\alpha_2,\beta_2)} (\mathfrak{f}_3, A_3) \xrightarrow{(\alpha_3,\beta_3)} (\mathfrak{f}_4, A_4)$ *is* fse*. Then* (α_1, β_1) *is epic iff* (α_3, β_3) *is monic.*

Proof. Let (α_1, β_1) be epic. Since, the sequence $(\mathfrak{f}_1, A_1) \xrightarrow{(\alpha_1,\beta_1)} (\mathfrak{f}_2, A_2) \xrightarrow{(\alpha_2,\beta_2)} (\mathfrak{f}_3, A_3) \xrightarrow{(\alpha_3,\beta_3)} (\mathfrak{f}_4, A_4)$ is fse, therefore, from Proposition 4.4, $\mathrm{Im}(\alpha_1, \beta_1) = (\mathfrak{f}_2, A_2) = \mathrm{Ker}(\alpha_2, \beta_2)$. This implies $(\mathfrak{f}_3, A_3) = 1$, and hence from Proposition 4.4 (α_3, β_3) is monic.
Conversely, if (α_3, β_3) is monic, then from Proposition 4.4, $\mathrm{Ker}(\alpha_3, \beta_3) = 1$. This implies $\mathrm{Im}(\alpha_2, \beta_2) = \mathrm{Ker}(\alpha_3, \beta_3) = 1$ which implies, $(\mathfrak{f}_1, A_1) \xrightarrow{(\alpha_1,\beta_1)} (\mathfrak{f}_2, A_2) \xrightarrow{(\alpha_2,\beta_2)} 1$ is fse. Hence, by Proposition 4.4, (α_1, β_1) is epic. This completes the proof. □

We conclude the paper with the following interesting result on fse $X-$modules.

Theorem 4.8. *Let* $(\mathfrak{f}_i, A_i)$ *be* $fsX-$*modules over the* $X-$*modules* M_i*, where* $i = 1, 2, 3, 4, 5$*, respectively. If* $(\mathfrak{f}_1, A_1) \xrightarrow{(\alpha_1,\beta_1)} (\mathfrak{f}_2, A_2) \xrightarrow{(\alpha_2,\beta_2)} (\mathfrak{f}_3, A_3) \xrightarrow{(\alpha_3,\beta_3)} (\mathfrak{f}_4, A_4) \xrightarrow{(\alpha_4,\beta_4)} (\mathfrak{f}_5, A_5)$ *is an* fse *sequence of* $X-$*modules, with* (α_1, β_1) *is epic and* (α_4, β_4) *is monic. Then* $(\mathfrak{f}_3, A_3) = 1$.

Proof. Let (α_1, β_1) be epic. Then from Lemma 4.7, (α_3, β_3) is monic. Similarly, since (α_4, β_4) is monic, therefore, from Lemma 4.7, (α_2, β_2) is epic. Finally, from Lemma 4.6, $(\mathfrak{f}_3, A_3) = 1$. This completes the proof. $\square$

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