Punjab University
Journal of Mathematics (ISSN 1016-2526)
Vol. 51(3)(2019) pp. 47-59

Two High Order Iterative Methods for Roots of Nonlinear Equations

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Received: 14 April, 2018 / Accepted: 09 July, 2018 / Published online: 26 December, 2018

Abstract. This study suggests new iterative methods, based on the conventional Newton's method, to obtain the numerical solutions of nonlinear equations. We prove that our methods include five and ten orders of convergence. Also, the convergence behavior and comparison with an existing results of the proposed schemes are investigated. Numerical experiments demonstrate that the proposed schemes are able to attain up to the better accuracy than some classical methods, while still significantly reducing the total number of calculations and iterations.

AMS (MOS) Subject Classification Codes: 65H05; 49M15; 41A25
Key Words: Nonlinear equation, Iterative method, Newton's method, Convergence order.

1. Introduction

Finding the roots of nonlinear equations is very important in numerical analysis and has many applications in science and engineering [11, 15, 18, 19]. Numerical solution of many equations such as ODEs, PDEs and integral equations yield to solve of nonlinear equations [3, 14]. In this paper, we introduce two new iterative methods to find a simple root x^* of a nonlinear equation $f(x) = 0$, where $f : D \subset \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval D.

One of the well-known methods for solving nonlinear equations is the classical Newton's (CN) method given by

$$x_{n+1} \quad = \quad x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0, 1, 2, \ldots. \tag{1. 1}$$

This method converges quadratically in some neighborhoods of a simple root x^* of f.

The first attempts for classifying iterative methods were done by Traub [19]. He suggested a third-order iterative method. Jarratt [9, 10], proposed a family of methods consisting of two points, two steps, costing one function, two derivative evaluations per iteration and one parameter to reach order of convergence four. Jain [8], discussed a fifth-order implicit method that uses the information of one function and three derivative evaluations per iteration. In recent years, many authors developed high order iterative methods and investigated the convergence analysis of them for solving nonlinear equations, see [1, 2, 4, 6, 11, 15, 16, 18] and the references therein. We suggest and analyze a three steps, fifth-order, and a four steps, tenth-order iterative methods for solving nonlinear equations.

The paper is structured as follows. In the next section, we give the new iterative formulas and establish the convergence orders of these approaches. In Section 3, different numerical tests confirm the theoretical results and allow us to compare these variants with classical methods.

2. MAIN RESULTS

Firstly, we introduce new fifth-order (FO) iterative method as follows,

$$\begin{aligned} \vartheta_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ \mu_n &= x_n + \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= x_n - \frac{(x_n-\vartheta_n)f(x_n)^2(f(\vartheta_n)+f(\mu_n))}{f(x_n)^2(f(\mu_n)-f(\vartheta_n))-4f(x_n)f(\vartheta_n)^2-6f(\vartheta_n)^3}, \qquad n=0,1,2,\ldots. \end{aligned} \tag{2. 2}$$

We can state the following convergence theorem for the three steps method (2. 2).

Theorem 2.1. *Assume that the function $f: D \subset \mathbb{R} \to \mathbb{R}$ has a simple root $x^* \in D$, where D is an open interval. If $f(x)$ to be sufficiently smooth in the neighborhood of the root x^*, then the order of convergence of method* (2. 2) *is equal to five.*

Proof. We denote the error in each iteration with $e_n = x_n - x^*$. Then by expanding $f(x)$ and $f'(x)$ in Taylor series about x^*, we have

$$f(x_n) = f'(x^*)\left(e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)\right), \tag{2. 3}$$

and so

$$f'(x_n) = f'(x^*)\left(1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + O(e_n^6)\right), \tag{2. 4}$$

where $c_n = \frac{1}{n!}\frac{f^{(n)}(x^*)}{f'(x^*)}$, $n \geq 2$.
Using (2. 2)–(2. 4), gives us

$$\begin{aligned} \frac{f(x_n)}{f'(x_n)} &= x_n - \vartheta_n = e_n - c_2e_n^2 + (-2c_3 + 2c_2^2)e_n^3 \\ &\quad +(-3c_4 + 3c_2c_3 + 2c_2(2c_3 - 2c_2^2))e_n^4 \\ &\quad +(-4c_5 + 4c_2c_4 + (3(2c_3 - 2c_2^2))c_3 \\ &\qquad +(2(3c_4 - 7c_2c_3 + 4c_2^3))c_2)e_n^5 + O(e_n^6), \end{aligned} \tag{2. 5}$$

in addition to

$$\begin{aligned} f(x_n)^2 &= (f'(x^*))^2\Big(e_n^2 + 2c_2e_n^3 + (2c_3 + c_2^2)e_n^4 \\ &\quad +(2c_4 + 2c_2c_3)e_n^5 + (2c_5 + 2c_2c_4 + c_3^2)e_n^6 + O(e_n^7)\Big), \end{aligned} \tag{2. 6}$$

and hence, for the first and second steps of the (2. 2), we attain

$$\begin{aligned}\vartheta_n &= x_n - \tfrac{f(x_n)}{f'(x_n)} = x^* + c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 \\ &\quad +(4c_5 - 10c_2c_4 - 6c_3^2 + 20c_3c_2^2 - 8c_2^4)e_n^5 + O(e_n^6),\end{aligned} \tag{2. 7}$$

and

$$\begin{aligned}\mu_n &= x_n + \tfrac{f(x_n)}{f'(x_n)} = x_n + e_n - c_2e_n^2 + (-2c_3 + 2c_2^2)e_n^3 \\ &\quad +(-3c_4 + 7c_2c_3 - 4c_2^3)e_n^4 \\ &\quad +(-4c_5 + 10c_2c_4 + 6c_3^2 - 20c_3c_2^2 + 8c_2^4)e_n^5 + O(e_n^6).\end{aligned} \tag{2. 8}$$

Expanding $f(\vartheta_n)$ and $f(\mu_n)$ in the Taylor's series about x^*, using (2. 7) and (2. 8), we have

$$\begin{aligned}f(\vartheta_n) &= f'(x^*)\Big(c_2e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 5c_2^3)e_n^4 \\ &\quad +(4c_5 - 10c_2c_4 - 6c_3^2 + 24c_3c_2^2 - 12c_2^4)e_n^5 + O(e_n^6)\Big),\end{aligned} \tag{2. 9}$$

and

$$\begin{aligned}f(\mu_n) &= f'(x^*)\Big(2e_n + 3c_2e_n^2 + (-2c_2^2 + 6c_3)e_n^3 + (-13c_2c_3 + 5c_2^3 + 13c_4)e_n^4 \\ &\quad +(-34c_2c_4 + 42c_3c_2^2 - 12c_2^4 + 28c_5 - 18c_3^2)e_n^5 + O(e_n^6)\Big).\end{aligned} \tag{2. 10}$$

Now using (2. 9) and (2. 10), we get

$$\begin{aligned}f(\vartheta_n) + f(\mu_n) &= f'(x^*)\Big(2e_n + 4c_2e_n^2 + (8c_3 - 4c_2^2)e_n^3 + (16c_4 - 20c_2c_3 + 10c_2^3)e_n^4 \\ &\quad +(32c_5 - 44c_2c_4 - 24c_3^2 + 66c_3c_2^2 - 24c_2^4)e_n^5 + O(e_n^6)\Big)\end{aligned} \tag{2. 11}$$

and

$$\begin{aligned}f(\mu_n) - f(\vartheta_n) &= f'(x^*)\Big(2e_n + 2c_2e_n^2 + 4c_3e_n^3 + (10c_4 - 6c_2c_3)e_n^4 \\ &\quad +(24c_5 - 24c_2c_4 - 12c_3^2 + 18c_3c_2^2)e_n^5 + O(e_n^6)\Big).\end{aligned} \tag{2. 12}$$

Furthermore, from (2. 9), we have

$$\begin{aligned}f(\vartheta_n)^2 &= (f'(x^*))^2\Big(c_2^2e_n^4 - 4c_2(-c_3 + c_2^2)e_n^5 + 2\left(3c_2c_4 - 11c_2^2c_3 + 7c_2^4 + 2c_3^2\right)e_n^6 \\ &\quad -4\left(-2c_2c_5 + 8c_4c_2^2 + 10c_2c_3^2 - 24c_3c_2^3 + 11c_2^5 - 3c_4c_3\right)e_n^7 + O(e_n^8)\Big), \\ f(\vartheta_n)^3 &= (f'(x^*))^3\Big(c_2^3e_n^6 - 6c_2^2(-c_3 + c_2^2)e_n^7 + O(e_n^8)\Big).\end{aligned} \tag{2. 13}$$

Applying (2. 5), (2. 6) and (2. 11), we obtain

$$\begin{aligned}&(x_n - \vartheta_n)f(x_n)^2(f(\vartheta_n) + f(\mu_n)) \\ &\quad = (f'(x^*))^3\Big(2e_n^4 + 6c_2e_n^5 + (2c_2^2 + 8c_3)e_n^6 + (14c_4 + 8c_2^3 \\ &\qquad -6c_2c_3)e_n^7 + (28c_5 - 24c_2c_4 - 18c_3^2 + 54c_3c_2^2 - 18c_2^4)e_n^8 + O(e_n^9)\Big).\end{aligned} \tag{2. 14}$$

From (2. 3), (2. 6), (2. 12) and (2. 13), we get

$$\begin{aligned} &f(x_n)^2(f(\mu_n)-f(\vartheta_n))-4f(x_n)f(\vartheta_n)^2-6f(\vartheta_n)^3 \\ &\quad=(f'(x^*))^3\Big(2e_n^3+6c_2e_n^4+(2c_2^2+8c_3)e_n^5+(-6c_2c_3+14c_4+8c_2^3)e_n^6 \\ &\qquad+(28c_5-18c_3^2-4c_2^4-20c_2c_4+46c_3c_2^2)e_n^7+O(e_n^8)\Big). \end{aligned} \tag{2. 15}$$

Finally, by using (2. 14) and (2. 15), we obtain the error relation as

$$e_{n+1}=x_{n+1}-x^*=c_2(-4c_2c_3+7c_2^3+2c_4)e_n^5+O(e_n^6).$$

This means that the method defined by (2. 2) is of the fifth-order. □

Secondly, by combining the classical Newton's method (1. 1) and FO method (2. 2), we construct the following ten order (TO) iterative method

$$\begin{aligned} &\vartheta_n=x_n-\frac{f(x_n)}{f'(x_n)},\quad \eta_n=\vartheta_n-\frac{f(\vartheta_n)}{f'(\vartheta_n)},\quad \lambda_n=\vartheta_n+\frac{f(\vartheta_n)}{f'(\vartheta_n)}, \\ &x_{n+1}=\vartheta_n-\frac{(\vartheta_n-\eta_n)f(\vartheta_n)^2(f(\eta_n)+f(\lambda_n))}{f(\vartheta_n)^2(f(\lambda_n)-f(\eta_n))-4f(\vartheta_n)f(\eta_n)^2-6f(\eta_n)^3},\qquad n=0,1,2,\ldots. \end{aligned} \tag{2. 16}$$

Theorem 2.2. *Let $x^*\in D$ be a simple zero of the sufficiently differentiable function $f: D\subset\mathbb{R}\to\mathbb{R}$, for an open interval D. If x_0 is sufficiently close to the root x^*, then four step iterative method* (2. 16) *has order of convergence* 10*.*

Proof. We expand $f(x)$ and $f'(x)$ in Taylor series about x^*, and up to order 10, thus

$$\begin{aligned} f(x_n) &= f'(x^*)\left(e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+c_5e_n^5+c_6e_n^6\right. \\ &\quad\left.+c_7e_n^7+c_8e_n^8+c_9e_n^9+c_{10}e_n^{10}+O(e_n^{11})\right), \end{aligned}$$

and

$$\begin{aligned} f'(x_n) &= f'(x^*)\left(1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+5c_5e_n^4+6c_6e_n^5\right. \\ &\quad\left.+c_6e_n^6+c_7e_n^7+c_8e_n^8+c_9e_n^9+O(e_n^{10})\right), \end{aligned}$$

therefore

$$\begin{aligned} \vartheta_n &= e_n-\tfrac{f(x_n)}{f'(x_n)} \\ &= c_2e_n^2+\left(-2c_2^2+2c_3\right)e_n^3+\left(4c_2^3-7c_2c_3+3c_4\right)e_n^4 \\ &\quad+\left(-8c_2^4+20c_3c_2^2-6c_3^2-10c_2c_4+4c_5\right)e_n^5 \\ &\quad+\left(16c_2^5-52c_2^3c_3+28c_2^2c_4-17c_3c_4+c_2(33c_3^2-13c_5)+5c_6\right)e_n^6 \\ &\quad-2\big(16c_2^6-64c_2^4c_3-9c_3^3+36c_2^3c_4+6c_4^2+9c_2^2(63c_3^2-2c_5) \\ &\qquad+11c_3c_5-2c_2(23c_3c_4-4c_6)-3c_7\big)e_n^7 \\ &\quad+\Big(64c_2^7-304c_2^5c_3+176c_2^4c_4+4c_2^3(102c_3^2-23c_5)-4c_2^2(87c_3c_4-11c_6) \\ &\qquad-c_2(135c_3^3-118c_3c_5-64c_4^2+19c_7) \\ &\qquad+3c_3(25c_3c_4-9c_6)-31c_4c_5+7c_8\Big)e_n^8 \end{aligned}$$

$$
\begin{aligned}
&+\Big(-128c_2^8+704c_2^6c_3-416c_2^5c_4+c_2^4(-1200c_3^2+224c_5)\\
&\quad +c_2^3(-112c_6+1120c_3c_4)+c_2^2(648c_3^3-240c_4^2-444c_3c_5+52c_7)\\
&\quad +c_2(-558c_3^2c_4+144c_3c_6+164c_4c_5-22c_8)-54c_3^4\\
&\quad +96c_3^2c_5+c_3(104c_4^2-32c_7)-38c_4c_6-20c_5^2+8c_9\Big)e_n^9\\
&+\Big(256c_2^9-1600c_2^7c_3+960c_2^6c_4+c_2^5(3312c_3^2-528c_5)\\
&\quad +c_2^4(-3280c_3c_4272c_6)+c_2^3(-2520c_3^3+1424c_3c_5+768c_4^2-132c_7)\\
&\quad +c_2^2(2664c_3^2c_4-540c_3c_6-612c_4c_5+60c_8)\\
&\quad +c_2\left(513c_3^4-711c_3^2c_5+c_3(-768c_4^2+170c_7)+200c_4c_6+105c_5^2-25c_9\right)\\
&\quad -297c_3^3c_4+117c_3^2c_6+c_3(266c_4c_5-37c_8)+48c_4^3\\
&\quad -45c_4c_7-49c_5c_6+9c_{10}\Big)e_n^{10}+O(e_n^{11}).
\end{aligned}
\tag{2. 17}
$$

By expanding $f(\vartheta_n)$ around x^*, we get

$$
\begin{aligned}
f(\vartheta_n) \;=\;& f'(x^*)\Big(c_2e_n^2+(-2c_2^2+2c_3)e_n^3+(5c_2^3-7c_2c_3+3c_4)e_n^4\\
&+(-12c_2^4+24c_2^2c_3-10c_2c_4-6c_3^2+4c_5)e_n^5\\
&+(28c_2^5-73c_2^3c_3+34c_2^2c_4+c_2(37c_3^2-13c_5)-17c_3c_4+5c_6)e_n^6\\
&+\big(-64c_2^6+206c_2^4c_3-104c_2^3c_4+c_2^2(-160c_3^2+44c_5)\\
&\quad +c_2(104c_3c_4-16c_6)+18c_3^3-22c_3c_5-12c_4^2+6c_7\big)e_n^7\\
&+\big(144c_2^7-552c_2^5c_3+297c_2^4c_4+c_2^3(582c_3^2-134c_5)\\
&\quad +c_2^2(-455c_3c_4+54c_6)+c_2(-147c_3^3+134c_3c_5+73c_4^2-19c_7)\\
&\quad +75c_3^2c_4-27c_3c_6-31c_4c_5+7c_8\big)e_n^8\\
&+(-320c_2^8+1424c_2^6c_3-808c_2^5c_4+c_2^4(-1904c_3^2+380c_5)\\
&\quad +c_2^3(1686c_3c_4-164c_6)+c_2^2(774c_3^3-584c_3c_5-324c_4^2\\
&\quad +64c_7)+c_2(-626c_3^2c_4+164c_3c_6+188c_4c_5-22c_8)-46c_3^4\\
&\quad +96c_3^2c_5+c_3(104c_4^2-32c_7)-38c_4c_6-20c_5^2+8c_9)e_n^9\\
&+(704c_2^9-3568c_2^7c_3+2120c_2^6c_4+c_2^5(5792c_3^2-1023c_5)\\
&\quad +c_2^4(-5636c_3c_4+464c_6)+c_2^3(-3294c_3^3+2147c_3c_5+1224c_4^2\\
&\quad -194c_7)+c_2^2(3393c_3^2c_4-713c_3c_6-832c_4c_5+74c_8)+c_2(465c_3^4\\
&\quad -799c_3^2c_5-891c_3c_4^2+194c_3c_7+230c_4c_6+121c_5^2-25c_9)\\
&\quad -261c_3^3c_4+117c_3^2c_6+c_3(266c_4c_5-37c_8)+48c_4^3-45c_4c_7\\
&\quad -49c_5c_6+9c_{10})e_n^{10}+O(e_n^{11})\Big),
\end{aligned}
\tag{2. 18}
$$

also

$$
\begin{aligned}
f'(\vartheta_n) \;=\;& f'(x^*)\Big(1+2c_2^2e_n^2-4c_2(c_2^2-c_3)e_n^3+(8c_2^4-11c_2^2c_3+6c_2c_4)e_n^4\\
&+(-16c_2^5+28c_2^3c_3-20c_2^2c_4+8c_2c_5)e_n^5\\
&+(32c_2^6-68c_2^4c_3+60c_2^3c_4-26c_2^2c_5+c_2(-16c_3c_4+10c_6)+12c_3^3)e_n^6\\
&+(-64c_2^7+160c_2^5c_3-168c_2^4c_4+72c_2^3c_5+c_2^2(112c_3c_4-32c_6)\\
&\quad +c_2(-84c_3^3-20c_3c_5-24c_4^2+12c_7)+36c_3^2c_4)e_n^7\\
&+(128c_2^8-368c_2^6c_3+448c_2^5c_4-179c_2^4c_5+c_2^3(-516c_3c_4+88c_6)\\
&\quad +c_2^2(387c_3^3+110c_3c_5+164c_4^2-38c_7)+c_2(-150c_3^2c_4-24c_3c_6\\
&\quad -62c_4c_5+14c_8)-72c_3^4+48c_3^2c_5+27c_3c_4^2)e_n^8+O(e_n^9)\Big).
\end{aligned}
\tag{2. 19}
$$

Using (2. 17)–(2. 19), we have

$$\begin{aligned}
\eta_n &= \vartheta_n - \tfrac{f(\vartheta_n)}{f'(\vartheta_n)}\\
&= c_2^3e_n^4 + (-4c_2^4 + 4c_2^2c_3)e_n^5 + (10c_2^5 - 20c_2^3c_3 + 6c_2^2c_4 + 4c_2c_3^2)e_n^6\\
&\quad +(-20c_2^6 + 60c_2^4c_3 - 32c_2^3c_4 + c_2^2(-28c_3^2 + 8c_5) + 12c_2c_3c_4)e_n^7\\
&\quad +(36c_2^7 - 141c_2^5c_3 + 105c_2^4c_4 + c_2^3(105c_3^2 - 42c_5)\\
&\qquad +c_2^2(-98c_3c_4 + 10c_6) + c_2(16c_3c_5 + 9c_4^2))e_n^8\\
&\quad +\big(-64c_2^8 + 296c_2^6c_3 - 276c_2^5c_4 + c_2^4(-292c_3^2 + 132c_5)\\
&\qquad +c_2^3(444c_3c_4 - 52c_6) + c_2^2(-16c_3^3 - 128c_3c_5 - 84c_4^2 + 12c_7)\\
&\qquad +c_2(-32c_3^2c_4 + 20c_3c_6 + 24c_4c_5) + 16c_3^4\big)e_n^9\\
&\quad +\big(120c_2^9 - 612c_2^7c_3 + 654c_2^6c_4 + c_2^5(714c_3^2 - 318c_5)\\
&\qquad +c_2^4(-1542c_3c_4 + 162c_6) + c_2^3(144c_3^3 + 540c_3c_5 + 426c_4^2 - 62c_7)\\
&\qquad +c_2^2(336c_3^2c_4 - 158c_3c_6 - 220c_4c_5 + 14c_8) + c_2(-204c_3^4 - 40c_3^2c_5\\
&\qquad +c_3(-96c_4^2 + 24c_7) + 30c_4c_6 + 16c_5^2) + 72c_3^3c_4\big)e_n^{10} + O(e_n^{11}),
\end{aligned} \tag{2. 20}$$

and

$$\begin{aligned}
\lambda_n &= \vartheta_n + \tfrac{f(\vartheta_n)}{f'(\vartheta_n)}\\
&= 2c_2e_n^2 + (-4c_2^2 + 4c_3)e_n^3 + (7c_2^3 - 14c_2c_3 + 6c_4)e_n^4\\
&\quad +(-12c_2^4 + 36c_2^2c_3 - 20c_2c_4 - 12c_3^2 + 8c_5)e_n^5\\
&\quad +\big(22c_2^5 - 84c_2^3c_3 + 50c_2^2c_4 + c_2(62c_3^2 - 26c_5) - 34c_3c_4 + 10c_6\big)e_n^6\\
&\quad +(-44c_2^6 + 196c_3c_2^4 - 112c_4c_2^3 + (-224c_3^2 + 64c_5)c_2^2\\
&\qquad +c_2(172c_3c_4 - 32c_6) + 36c_3^3 - 44c_3c_5 - 24c_4^2 + 12c_7)e_n^7\\
&\quad +(92c_2^7 - 467c_2^5c_3 + 247c_2^4c_4 + c_2^3(711c_3^2 - 142c_5)\\
&\qquad +c_2^2(-598c_3c_4 + 78c_6) + c_2(-270c_3^3 + 220c_3c_5 + 119c_4^2 - 38c_7)\\
&\qquad +150c_3^2c_4 - 54c_3c_6 - 62c_4c_5 + 14c_8)e_n^8\\
&\quad +\big(-192c_2^8 + 1112c_2^6c_3 - 556c_2^5c_4 + c_2^4(-2108c_3^2 + 316c_5)\\
&\qquad +c_2^3(1796c_3c_4 - 172c_6) + c_2^2(1312c_3^3 - 760c_3c_5 - 396c_4^2 + 92c_7)\\
&\qquad +c_2(-1084c_3^2c_4 + 268c_3c_6 + 304c_4c_5 - 44c_8) - 124c_3^4 + 192c_3^2c_5\\
&\qquad +c_3(208c_4^2 - 64c_7) - 76c_4c_6 - 40c_5^2 + 16c_9\big)e_n^9\\
&\quad +(392c_2^9 - 2588c_2^7c_3 + 1266c_2^6c_4 + c_2^5(5910c_3^2 - 738c_5)\\
&\qquad +c_2^4(-5018c_3c_4 + 382c_6) + c_2^3(-5184c_3^3 + 2308c_3c_5 + 1110c_4^2 - 202c_7)\\
&\qquad +c_2^2(4992c_3^2c_4 - 922c_3c_6 - 1004c_4c_5 + 106c_8) + c_2(1230c_3^4 - 1382c_3^2c_5\\
&\qquad +c_3(-1440c_4^2 + 316c_7) + 370c_4c_6 + 194c_5^2 - 50c_9) - 666c_3^3c_4 + 234c_3^2c_6\\
&\qquad +c_3(532c_4c_5 - 74c_8) + 96c_4^3 - 90c_4c_7 - 98c_5c_6 + 18c_{10})e_n^{10} + O(e_n^{11}).
\end{aligned} \tag{2. 21}$$

Taylor expansions of $f(\eta_n)$ and $f(\lambda_n)$ around x^* are given as

$$\begin{aligned}
f(\eta_n) &= f'(x^*)\Big(c_2^3e_n^4 + (-4c_2^4 + 4c_2^2c_3)e_n^5 + (10c_2^5 - 20c_2^3c_3 + 6c_2^2c_4 + 4c_2c_3^2)e_n^6\\
&\quad +(-20c_2^6 + 60c_2^4c_3 - 32c_2^3c_4 + c_2^2(-28c_3^2 + 8c_5) + 12c_2c_3c_4)e_n^7\\
&\quad +(37c_2^7 - 141c_2^5c_3 + 105c_2^4c_4 + c_2^3(105c_3^2 - 42c_5)\\
&\qquad +c_2^2(-98c_3c_4 + 10c_6) + c_2(16c_3c_5 + 9c_4^2))e_n^8\\
&\quad +(-72c_2^8 + 304c_2^6c_3 - 276c_2^5c_4 + c_2^4(-292c_3^2 + 132c_5)\\
&\qquad +c_2^3(444c_3c_4 - 52c_6) + c_2^2(-16c_3^3 - 128c_3c_5 - 84c_4^2 + 12c_7)\\
&\qquad +c_2(-32c_3^2c_4 + 20c_3c_6 + 24c_4c_5) + 16c_3^4)e_n^9
\end{aligned}$$

$$
\begin{aligned}
&+(156c_2^9-684c_2^7c_3+666c_2^6c_4+c_2^5(738c_3^2-318c_5)\\
&\quad+c_2^4(-1542c_3c_4+162c_6)+c_2^3(144c_3^3+540c_3c_5+426c_4^2-62c_7)\\
&\quad+c_2^2(336c_3^2c_4-158c_3c_6-220c_4c_5+14c_8)+c_2(-204c_3^4-40c_3^2c_5\\
&\quad+c_3(-96c_4^2+24c_7)+30c_4c_6+16c_5^2)+72c_3^3c_4)e_n^{10}+O(e_n^{11})\Big),
\end{aligned}
\tag{2. 22}
$$

and

$$
\begin{aligned}
f(\lambda_n) \;=\;& f'(x^*)\Big(2c_2e_n^2+(-4c_2^2+4c_3)e_n^3+(11c_2^3-14c_2c_3+6c_4)e_n^4\\
&+(-28c_2^4+52c_2^2c_3-20c_2c_4-12c_3^2+8c_5)e_n^5\\
&+(66c_2^5-164c_2^3c_3+74c_2^2c_4+c_2(78c_3^2-26c_5)-34c_3c_4+10c_6)e_n^6\\
&+(-148c_2^6+460c_2^4c_3-240c_2^3c_4+c_2^2(-336c_3^2+96c_5)\\
&\quad+c_2(220c_3c_4-32c_6)+36c_3^3-44c_3c_5-24c_4^2+12c_7)e_n^7\\
&+\big(325c_2^7-1203c_2^5c_3+707c_2^4c_4+c_2^3(1179c_3^2-310c_5)\\
&\quad+c_2^2(-990c_3c_4+118c_6)+c_2(-270c_3^3+284c_3c_5+155c_4^2-38c_7)\\
&\quad+150c_3^2c_4-54c_3c_6-62c_4c_5+14c_8\big)e_n^8\\
&+(-712c_2^8+3040c_2^6c_3-1956c_2^5c_4+c_2^4(-3716c_3^2+892c_5)\\
&\quad+c_2^3(3748c_3c_4-380c_6)+c_2^2(1280c_3^3-1272c_3c_5-732c_4^2+140c_7)\\
&\quad+c_2(-1212c_4c_3^2+348c_3c_6+400c_4c_5-44c_8)-60c_3^4\\
&\quad+192c_3^2c_5+c_3(208c_4^2-64c_7)-76c_4c_6-40c_5^2+16c_9)e_n^9\\
&+(1564c_2^9-7562c_2^7c_3+5202c_2^6c_4+c_2^5(11086c_3^2-2342c_5)\\
&\quad+c_2^4(-12870c_3c_4+1090c_6)+c_2^3(-5040c_3^3+4660c_3c_5\\
&\quad+2970c_4^2-450c_7)+c_2^2(6576c_3^2c_4-1554c_3c_6-1884c_4c_5+162c_8)\\
&\quad+c_2(414c_3^4-1542c_3^2c_5+c_3(-1824c_4^2+412c_7)+490c_4c_6\\
&\quad+258c_5^2-50c_9)-378c_3^3c_4+234c_3^2c_6+c_3(532c_4c_5-74c_8)96c_4^3\\
&\quad-90c_4c_7-98c_5c_6+18c_{10})e_n^{10}+O(e_n^{11})\Big).
\end{aligned}
\tag{2. 23}
$$

Using the above relations, we obtain

$$
\begin{aligned}
&(\vartheta_n-\eta_n)f(\vartheta_n)^2(f(\eta_n)+f(\lambda_n))\\
&\quad=(f'(x^*))^3\Big(2c_2^4e_n^8+(-16c_2^5+16c_2^3c_3)e_n^9\\
&\qquad+(86c_2^6-152c_2^4c_3+24c_2^3c_4+48c_2^2c_3^2)e_n^{10}\\
&\qquad+\theta_1e_n^{11}+\theta_2e_n^{12}+\theta_3e_n^{13}+\theta_4e_n^{14}+\theta_5e_n^{15}+\theta_6e_n^{16}+O(e_n^{17})\Big),
\end{aligned}
\tag{2. 24}
$$

where

$$
\theta_1 \;=\; -380c_2^7+940c_2^5c_3-224c_2^4c_4+c_2^3(-576c_3^2+32c_5)+144c_2^2c_3c_4+64c_2c_3^3,
$$

$$
\begin{aligned}
\theta_2 \;=\;& 1482c_2^8-4682c_2^6c_3+1370c_2^5c_4+c_2^4(4260c_3^2-296c_5)+c_2^3(-1696c_3c_4\\
&+40c_6)+c_2^2(-1088c_3^3+192c_3c_5+108c_4^2)+288c_2c_3^2c_4+32c_3^4,
\end{aligned}
$$

$$
\begin{aligned}
\theta_3 \;=\;& -5288c_2^9+20288c_2^7c_3-6780c_2^6c_4+c_2^5(-24564c_3^2+1800c_5)\\
&+c_2^4(12400c_3c_4-368c_6)+c_2^3(10240c_3^3-2240c_3c_5-1248c_4^2+48c_7)\\
&+c_2^2(240c_3c_6-4800c_3^2c_4+288c_4c_5)+c_2(-1024c_3^4+384c_3^2c_5\\
&+432c_3c_4^2)+192c_3^3c_4,
\end{aligned}
$$

$$
\theta_4 \;=\; 17648c_2^{10}-79570c_2^8c_3+29290c_2^7c_4+c_2^6(120594c_3^2-8870c_5)
$$

$$+c_2^5(-71046c_3c_4+2230c_6)+c_2^4(-71080c_3^3+16280c_3c_5+9020c_4^2-440c_7)+c_2^3(44640c_3^2c_4-2784c_3c_6-3296c_4c_5+56c_8)+c_2^2(13760c_3^4-6336c_3^2c_5+c_3(-7056c_4^2+288c_7)+360c_4c_6+192c_5^2)+c_2(480c_3^2c_6-6016c_3^3c_4+216c_4^3+1152c_3c_4c_5)-384c_3^5+256c_3^3c_5+432c_3^2c_4^2,$$

$$\theta_5 = -55920c_2^{11}+289436c_2^9c_3-114804c_2^8c_4+c_2^7(-527712c_3^2+38168c_5)+c_2^6(347876c_3c_4-10960c_6)+c_2^5(406132c_3^3-92868c_3c_5-51360c_4^2+2660c_7)+c_2^4(-307920c_3^2c_4+20160c_3c_6+23680c_4c_5-512c_8)+c_2^3(-122400c_3^4+58560c_3^2c_5+c_3(64840c_4^2-3328c_7)-4096c_4c_6-2176c_5^2+64c_9)+c_2^2(79840c_3^3c_4-7872c_6c_3^2+c_3(-18624c_4c_5+336c_8)-3456c_4^3+432c_4c_7+480c_5c_6)+c_2(9792c_3^5-7936c_3^3c_5+c_3^2(-13248c_4^2+576c_7)+c_3(1440c_4c_6+768c_5^2)+864c_4^2c_5)-2816c_3^4c_4+320c_3^3c_6+1152c_5c_3^2c_4+432c_3c_4^3,$$

$$\theta_6 = 170030c_2^{12}-992690c_2^{10}c_3+418016c_2^9c_4+c_2^8(2117118c_3^2-148952c_5)+c_2^7(-1522332c_3c_4+47060c_6)+c_2^6(-2022238c_3^3+452876c_3c_5+250884c_4^2-13050c_7)+c_2^5(1755630c_3^2c_4-114690c_3c_6-134250c_4c_5+3090c_8)+c_2^4(846550c_3^4-402120c_5c_3^2+c_3(-444540c_4^2+24040c_7)+29320c_4c_6+15540c_5^2-584c_9)+c_2^3(-705840c_3^3c_4+72480c_3^2c_6+c_3(170080c_4c_5-3872c_8)-5408c_5c_6-4896c_4c_7+31380c_4^3+72c_{10})+c_2^2(-125280c_3^5+104640c_3^3c_5+c_3^2(173640c_4^2-9408c_7)+c_3(-23136c_4c_6-12288c_5^2+384c_9)+504c_4c_8-13680c_4^2c_5+576c_5c_7+300c_6^2)+c_2(70880c_3^4c_4-9856c_3^3c_6+c_3^2(-34944c_4c_5+672c_8)+c_3(-12960c_4^3+1920c_5c_6+1728c_4c_7)+1152c_4c_5^2+1080c_4^2c_6)+2880c_3^6-3712c_3^4c_5+c_3^3(-8256c_4^2+384c_7)+c_3^2(768c_5^2+1440c_4c_6)+1728c_3c_4^2c_5+162c_4^4,$$

and

$$f(\vartheta_n)^2(f(\lambda_n)-f(\eta_n))-4f(\vartheta_n)f(\eta_n)^2-6f(\eta_n)^3 = 2c_2^3e_n^6+(-12c_2^4+12c_2^2c_3)e_n^7+\big(54c_2^5-90c_3c_2^3+18c_4c_2^2+24c_2c_3^2\big)e_n^8+\big(-208c_2^6+480c_2^4c_3-132c_2^3c_4+c_2^2(-252c_3^2+24c_5)+72c_2c_3c_4+16c_3^3\big)e_n^9+(722c_2^7-2128c_2^5c_3+696c_2^4c_4+c_2^3(1692c_3^2-174c_5)+c_2^2(-738c_3c_4+30c_6)+c_2(-312c_3^3+96c_3c_5+54c_4^2)+72c_3^2c_4)e_n^{10}+\sigma_1e_n^{11}+\sigma_2e_n^{12}+\sigma_3e_n^{13}+\sigma_4e_n^{14}+O(e_n^{15})\Big), \tag{2. 25}$$

with

$$\sigma_1 = -2324c_2^8+8340c_2^6c_3-3072c_2^5c_4+c_2^4(-8968c_3^2+912c_5)+c_2^3(4896c_3c_4-216c_6)+c_2^2(2952c_3^3-972c_3c_5-540c_4^2+36c_7)+c_2(-1368c_3^2c_4+120c_3c_6+144c_4c_5)-144c_3^4+96c_5c_3^2+108c_3c_4^2,$$

$$\sigma_2 = 7072c_2^9-29900c_2^7c_3+12044c_2^6c_4+c_2^5(40740c_3^2-4008c_5)+c_2^4(1128c_6-25848c_3c_4)+c_2^3(-19884c_3^3+6408c_3c_5+3540c_4^2-258c_7)+c_2^2(12780c_3^2c_4-1206c_3c_6-1422c_4c_5+42c_8)+c_2(2544c_3^4-1800c_3^2c_5+c_3(-1998c_4^2+144c_7)+180c_4c_6+96c_5^2)-840c_3^3c_4+120c_3^2c_6+54c_4^3+288c_3c_4c_5,$$

$$\begin{aligned}\sigma_3 &= -20624c_2^10 + 100288c_2^8c_3 - 43324c_2^7c_4 + c_2^6(-165860c_3^2 + 15640c_5)\\ &+c_2^5(117608c_3c_4 - 4944c_6) + c_2^4(108800c_3^3 - 33680c_3c_5 - 18624c_4^2\\ &+1344c_7) + c_2^3(-85800c_3^2c_4 + 7920c_3c_6 + 9264c_4c_5 - 300c_8)\\ &+c_2^2(-24392c_3^4 + 16704c_5c_3^2 + c_3(18432c_4^2 - 1440c_7) - 1764c_4c_6\\ &-936c_5^2 + 48c_9) + c_2(14640c_3^3c_4 - 2232c_3^2c_6 + c_3(-5256c_4c_5 + 168c_8)\\ &-972c_4^3 + 216c_4c_7 + 240c_5c_6) + 864c_3^5 - 1104c_3^3c_5\\ &+c_3^2(-1836c_4^2 + 144c_7) + c_3(360c_4c_6 + 192c_5^2) + 216c_4^2c_5,\end{aligned}$$

$$\begin{aligned}\sigma_4 &= 58204c_2^{11} - 319634c_2^9c_3 + 146044c_2^8c_4 + c_2^7(622408c_3^2 - 55910c_5)\\ &+c_2^6(19250c_6 - 481210c_3c_4) + c_2^5(-516472c_3^3 + 152480c_3c_5\\ &+84912c_4^2 - 5880c_7) + c_2^4(470960c_3^2c_4 - 41512c_3c_6 - 48528c_4c_5\\ &+1560c_8) + c_2^3(170910c_3^4 - 111628c_3^2c_5 + c_3(-123408c_4^2 + 9432c_7)\\ &+11448c_4c_6 + 6060c_5^2 - 342c_9) + c_2^2(-140028c_3^3c_4 + 20628c_3^2c_6\\ &+c_3(48168c_4c_5 - 1674c_8) + 8856c_4^3 - 2106c_4c_7 - 2322c_5c_6 + 54c_{10})\\ &+c_2(-15632c_3^5 + 19104c_3^3c_5 + c_3^2(31572c_4^2 - 2664c_7)\\ &+c_3(-6516c_4c_6 - 3456c_5^2 + 192c_9) - 3834c_4^2c_5 + 252c_4c_8 + 288c_5c_7\\ &+150c_6^2) + 6192c_3^4c_4 - 1368c_3^3c_6 + c_3^2(-4824c_4c_5 + 168c_8)\\ &+c_3(-1782c_4^3 + 432c_4c_7 + 480c_5c_6) + 288c_4c_5^2 + 270c_4^2c_6.\end{aligned}$$

From (2. 17), (2. 24) and (2. 25), we get

$$e_{n+1} = x_{n+1} - x^* = c_2^6(7c_2^3 - 4c_2c_3 + 2c_4)e_n^{10} + O(e_n^{11}).$$

This means that the method defined by (2. 16) is of tenth–order, and completes the proof. □

Per iteration, the FO method requires 3 evaluations of f and 1 of f', whereas the TO method requires 4 evaluations of f and 2 of f'. We consider the definition of efficiency index as $p^{\frac{1}{w}}$ [7], where p is the order of the method and w is the number of function evaluations per iteration required by the method. If we assume that all evaluations have the same cost as function one, then the FO (2. 2) and TO (2. 16) methods have the efficiency indexes $\sqrt[4]{5} \approx 1.495$ and $\sqrt[6]{10} \approx 1.468$, respectively, which are better than $\sqrt{2} \approx 1.414$ of the Newton's method and $\sqrt[3]{3} \approx 1.442$ of the methods in [5].

3. Numerical results and conclusion

In this section, we employ the presented methods (2. 2) and (2. 16), to solve some nonlinear equations and compare them with the classical Newton's (CN) method (1. 1), NR2 method [17] for $\alpha = 1$, and LW(20) method [13] for $\beta_1 = \beta_2 = 1$. The considered test functions with corresponding simple roots are displayed in Table 1. Table 2 shows the results for all considered examples in Table 1. Fig. 1 shows a logarithmic error evaluation between two terms, $\log(|x_{n+1} - x_n|)$, for NR2, LW(20), FO and TO methods against the number of iterations n. Also, the logarithmic absolute values of the function, $\log(|f(x_n|)$, for NR2, LW(20), FO and TO methods with different values of n are presented in Fig. 2. Tables 3 and 4 are presented to report the results obtained of our method in comparison with the other methods for finding multiple roots of equations [12].

All numerical examples were performed in Matlab 7.14.0 (R2012a) using 300 digits floating point (digits := 300), and variable precision arithmetic. We have computed the root of each test function for three different initial guesses x_0, while the iterative schemes were stopped when $|x_{n+1} - x_n| + |f(x_n)| < 10^{-300}$. Note that we use $Div.$, when the iterative method is divergent. Also, IT denotes the number of iterations and in Table 2, the computational order of convergence ρ is approximated by means of [19],

$$\rho \approx \frac{\ln(|x_{n+1} - x_n|/|x_n - x_{n-1}|)}{\ln(|x_n - x_{n-1}|/|x_{n-1} - x_{n-2}|)}.$$

TABLE 1. Test functions with simple roots.

$f(x)$	x^*
$f_1(x) = xe^{x^2} - \cos(-x)$	0.588401776500996
$f_2(x) = x^{\frac{3}{2}} - 3x + 2$	$(\sqrt{3}+1)^2$
$f_3(x) = \cos(x) - \sinh(x) + 10$	2.896732637643961
$f_4(x) = e^{(-x^2+x+2)} - \cos(x-1) + x$	-1.159467272642035
$f_5(x) = \sin(x) + \cos(x) - i$	$\frac{1065}{452} - \frac{831}{1262}i$

TABLE 2. Numerical results for different methods with stoping criterium $|x_{n+1} - x_n| + |f(x_n)| < 10^{-300}$.

		CN	NR2($\alpha = 1$)	FO	TO
$f(x)$	x_0	(IT, ρ)	(IT, ρ)	(IT, ρ)	(IT, ρ)
f_1	–1.5	(15, 2.0000)	(8, 4.0000)	(7, 5.0000)	(6, 9.9996)
	0.75	(10, 2.0000)	(6, 4.0000)	(5, 4.9999)	(4, 9.9693)
	2.5	(18, 2.0000)	(10, 4.0000)	(9, 5.0000)	(7, 9.9997)
f_2	6.0	(10, 2.0000)	(7, 4.0000)	(6, 5.0000)	(4, 10.2031)
	7.0	(9, 2.0000)	(6, 4.0000)	(5, 5.0000)	(4, 10.0303)
	8.0	(9, 2.0000)	(7, 4.0000)	(5, 5.0000)	(4, 9.9664)
f_3	2.0	(11, 2.0000)	(6, 4.0000)	(6, 5.0002)	(4, 9.9801)
	3.0	(9, 2.0000)	(5, 4.0000)	(5, 5.0000)	(4, 9.9970)
	5.0	(12, 2.0000)	(7, 4.0000)	(6, 4.9999)	(5, 9.9976)
f_4	–1.0	(10, 2.0000)	(6, 4.0000)	(5, 4.9999)	(4, 9.9428)
	0.0	(11, 2.0000)	(8, 4.0000)	(6, 5.0000)	(5, 9.9999)
	0.25	(12, 2.0000)	(Div., —)	(6, 4.9996)	(5, 10.0188)
f_5	$5.0 - 2.5i$	(12, 2.0000)	(10, 4.0000)	(7, 5.0000)	(5, 9.9745)
	$6.0 - 1.0i$	(11, 2.0000)	(8, 4.0000)	(6, 5.0000)	(4, 9.5655)
	$7.0 - 7.0i$	(21, 2.0000)	(13, 4.0000)	(11, 5.0000)	(8, 10.2803)

TABLE 3. Test functions with multiple roots.

$f(x)$	Multiplicity	x^*
$f_6(x) = (x^3 + 4x^2 - 10)^4$	4	1.365230013414097
$f_7(x) = (\sin(x)^2 - x^2 + 1)^3$	3	1.404491648215341
$f_8(x) = (\cos(x) - x)^3$	3	0.739085133215161
$f_9(x) = (e^{-x^2} - x - 2)^5$	5	-1.980181055645692
$f_{10}(x) = (\cos(x) - \sqrt[3]{x^2})^5$	5	0.683090551943001

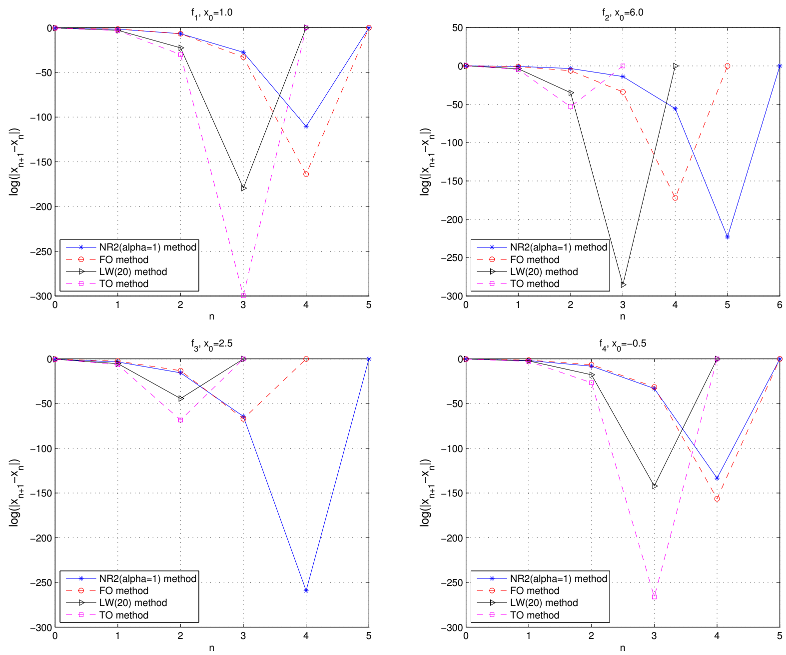

FIGURE 1. Logarithm of the absolute difference between consecutive terms for each of the four methods, versus number of iterations n.

TABLE 4. Numerical results for different methods with stoping criterium $|x_{n+1} - x_n| + |f(x_n)| < 10^{-300}$.

		IT			
$f(x)$	x_0	CN	NR2($\alpha = 1$)	FO	TO
f_6	0.5	85	54	44	32
	1	83	53	43	31
	2	87	55	45	32
f_7	0.75	58	39	31	26
	1	61	39	32	23
	2	64	41	34	28
f_8	0.5	61	39	32	23
	1	61	39	33	27
	1.5	64	40	34	28
f_9	–1	108	63	56	39
	–1.5	107	67	55	39
	–2.5	108	69	56	40
f_{10}	–0.5	111	70	59	41
	0.5	103	65	53	38
	1	106	67	55	39

It can be seen from the numerical results displayed in Tables 2, 4 and Figs 1, 2 that the proposed methods support the theoretical results proved in Section 2. Finally, we conclude

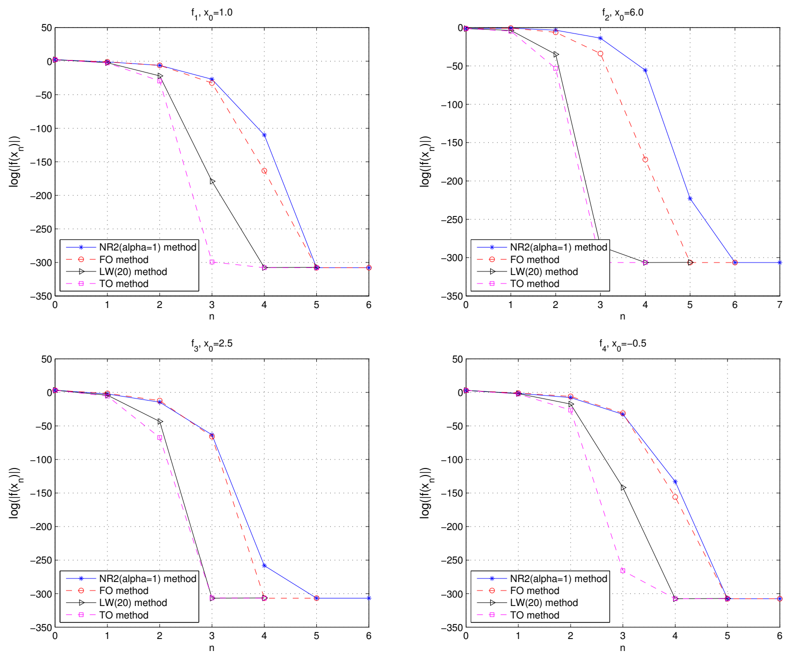

FIGURE 2. Logarithm of the residual, $\log |f(x_n)|$, achieved by each of the four methods, versus number of iterations n.

that the new iterative methods (2. 2) and (2. 16), presented in this paper, can compete with other efficient equation solvers, such as the Newton's method (1. 1), NR2 method [17] and LW(20) method [13].

4. ACKNOWLEDGMENTS

The authors would like to thank the referees for the valuable comments and for the suggestions to improve the readability of the paper.

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