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New Fejér and Hermite-Hadamard Type Inequalities for Differentiable p-Convex Mappings

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Abstract. In this paper, a new weighted identity involving a differentiable mapping and a non-negative p-symmetric mapping is established. By using the mathematical analysis techniques, some new integral inequalities of Hermite-Hadamard and Fejér type for differentiable p-convex functions are proved. A comparison of the established results is presented with the help of suitable graphs by using the software Mathematica.

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1. Introduction

Let $\lambda : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $\epsilon, \varepsilon \in I$ with $\epsilon < \varepsilon$, the double inequality

$$\lambda\left(\frac{\epsilon+\varepsilon}{2}\right) \leq \frac{1}{\varepsilon-\epsilon}\int_{\epsilon}^{\varepsilon} \lambda(\varkappa)\, d\varkappa \leq \frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2} \tag{1. 1}$$

is very famous in the theory of convex functions and is known as the Hermite-Hadamard inequality. The inequality (1. 1) is considered as a necessary and sufficient condition for a function λ to be convex over an interval I and it actually provides the bounds of the average value of a convex function.

In [9], Fejér gave a generalized version of (1. 1) while studying trigonometric polynomials. Fejér's original result reads as follows:

Consider the integral $\int_{\epsilon}^{\varepsilon} \lambda(\varkappa)\mu(\varkappa)d\varkappa$, where λ is a convex function in the interval (ϵ, ε), $\mu(\varkappa) > 0$ for $\varkappa$ in (ϵ, ε) and

$$\mu(\epsilon+\alpha) = \mu(\varepsilon-\alpha), \quad 0 \leq \alpha \leq \frac{1}{2}(\epsilon+\varepsilon).$$

Then

$$\lambda\left(\frac{\epsilon+\varepsilon}{2}\right)\int_{\epsilon}^{\varepsilon}\mu(\varkappa)d\varkappa\leq\int_{\epsilon}^{\varepsilon}\lambda(\varkappa)\mu(\varkappa)d\varkappa\leq\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_{\epsilon}^{\varepsilon}\mu(\varkappa)d\varkappa. \tag{1. 2}$$

It has been noticed that the theory of inequalities significantly depends on the theory of convexity. Since the theory of convexity plays an important role in the theory of inequalities and in the other areas of pure and applied mathematics, hence it has received a considerable attention by a number of researchers over the past few decades. Many mathematicians have tried to extend or to generalize the classical notion of convex sets and convex functions in several directions. As a consequence of the extensions and generalizations of the classical convexity, Hermite-Hadamard inequality (1. 1) and Fejér's inequality (1. 2) have been given different forms and numerous bounds related to the middle and leftmost, and middle and the rightmost terms in (1. 1) and (1. 2) have also been proved, see for instance [2]-[7] and [8]-[30].

One of the generalizations of the convex sets and convex functions, known as p-convex sets and p-convex functions, was introduced by Zhang in [31]. In the definitions of p-convex sets and p-convex functions given in [31], the number p is a positive odd integer or a fraction with numerator and denominator as positive odd integers and the p-convex functions are defined over an interval of the set of real numbers $\mathbb{R}$. The definitions of p-convex sets and p-convex functions were modified by İşcan in [15] by restricting the domain to be an interval of the set of positive real numbers so that p can be any non-zero real number. The class of p-convex functions introduced by İşcan contains both the class of classical convex functions and the class of harmonically convex functions that are defined over the set of positive real numbers.

In what follows we recall some basic definitions related to p-convex sets, p-convex functions, p-symmetric functions and related Hermite-Hadamard, and Fejér type inequalities for p-convex functions.

Definition 1.1. [31] *An interval $I\subset\mathbb{R}$ is p-convex if*

$$M_p\left(\varkappa,\beta;\alpha\right)=\left[\alpha\varkappa^p+\left(1-\alpha\right)\beta^p\right]^{\frac{1}{p}}\in I$$

for all $\varkappa$, $\beta\in I$ and $\alpha\in[0,1]$, where $p=2k+1$ or $p=\frac{n}{m}$, $n=2r+1$, $m=2s+1$, k, r, $s\in\mathbb{N}$.

Definition 1.2. [31] *Let I be a p-convex set. A function $\lambda:I\to\mathbb{R}$ is said to be p-convex function or λ is said to belong to the class $PC\left(I\right)$, if*

$$\lambda\left(M_p\left(\varkappa,\beta;\alpha\right)\right)\leq\alpha\lambda\left(\varkappa\right)+\left(1-\alpha\right)\lambda\left(\beta\right)$$

for all $\varkappa$, $\beta\in I$ and $\alpha\in[0,1]$.

Remark 1.3. *It is clear from the Definition 1.2 that the p-convex functions are the convex functions in the classical sense for $p=1$. Since $p=2k+1$ or $p=\frac{n}{m}$, $n=2r+1$, $m=2s+1$, k, r, $s\in\mathbb{N}$, this shows that $p\neq-1$. Hence the class $PC\left(I\right)$ does not contain the harmonic convex functions.*

Remark 1.4. [14] *If $I\subset(0,\infty)$ and $p\in\mathbb{R}\backslash\{0\}$, then*

$$M_p\left(\varkappa,\beta;\alpha\right)=\left[\alpha\varkappa^p+\left(1-\alpha\right)\beta^p\right]^{\frac{1}{p}}\in I$$

for all $\varkappa$, $\beta\in I$ and $\alpha\in[0,1]$.

Based on Remark 1.4, the following modification of p-convex functions was given in [14] by İşcan.

Definition 1.5. [15] *Let $I \subset (0,\infty)$ and $p \in \mathbb{R}\setminus\{0\}$. A function $\lambda : I \to \mathbb{R}$ is said to be p-convex function or λ is said to belong to the class $PC(I)$, if*

$$\lambda\left(M_p\left(\varkappa,\beta;\alpha\right)\right) \leq \alpha\lambda\left(\varkappa\right) + \left(1-\alpha\right)\lambda\left(\beta\right) \tag{1. 3}$$

for all $\varkappa$, $\beta \in I$ and $\alpha \in [0,1]$. If the inequality (1. 3) is reversed, then λ is said to be p-concave.

According to Definition 1.5, we get from the p-convexity the usual convexity and harmonic convexity when $p = 1$ and $p = -1$ of functions defined on $I \subset (0,\infty)$ respectively.

The following is the corrected version of a proposition given in [15].

Proposition 1.6. *Let $\lambda : I \subset (0,\infty) \to \mathbb{R}$ be a function and $p \in \mathbb{R}\setminus\{0\}$, then*

(1) *If λ is convex and nondecreasing, then λ is p-convex for $p \in (-\infty,0)\cup(0,1]$.*
(2) *If λ is p-convex and nondecreasing for $p \geq 1$, then λ is convex.*
(3) *If λ is p-concave and nondecreasing for $p \in (-\infty,0)\cup(0,1]$, then λ is concave.*
(4) *If λ is concave and nondecreasing, then λ is p-concave for $p \geq 1$.*
(5) *If λ is convex and nonincreasing, then λ is p-convex for $p \geq 1$.*
(6) *If λ is p-convex and nonincreasing for $p \in (-\infty,0)\cup(0,1]$, then λ is convex.*
(7) *If λ is p-concave and nonincreasing for $p \geq 1$, then λ is concave.*
(8) *If λ is concave and nonincreasing, then λ is p-concave for $p \in (-\infty,0)\cup(0,1]$.*

Proof. (1) Suppose that λ is convex and nondecreasing. For $p \in (-\infty,0)\cup(0,1]$, we have

$$\left[\alpha\varkappa^p + \left(1-\alpha\right)\beta^p\right]^{\frac{1}{p}} \leq \alpha\varkappa + \left(1-\alpha\right)\beta$$

for all $\varkappa$, $\beta \in I$ and $\alpha \in [0,1]$. Hence by using the convexity of λ, we have

$$\begin{aligned}\lambda\left(\left[\alpha\varkappa^p + \left(1-\alpha\right)\beta^p\right]^{\frac{1}{p}}\right) &\leq \lambda\left(\alpha\varkappa + \left(1-\alpha\right)\beta\right)\\ &\leq \alpha\lambda\left(\varkappa\right) + \left(1-\alpha\right) + \lambda\left(\beta\right)\end{aligned}$$

for all $\varkappa$, $\beta \in I$ and $\alpha \in [0,1]$. This shows that λ is p-convex.

(2) Suppose that λ is p-convex and nondecreasing for $p \geq 1$. For $p \geq 1$, we have

$$\alpha\varkappa + \left(1-\alpha\right)\beta \leq \left[\alpha\varkappa^p + \left(1-\alpha\right)\beta^p\right]^{\frac{1}{p}}$$

for all $\varkappa$, $\beta \in I$ and $\alpha \in [0,1]$. Hence by using the p-convexity of λ, we have

$$\begin{aligned}\lambda\left(\alpha\varkappa + \left(1-\alpha\right)\beta\right) &\leq \lambda\left(\left[\alpha\varkappa^p + \left(1-\alpha\right)\beta^p\right]^{\frac{1}{p}}\right)\\ &\leq \alpha\lambda\left(\varkappa\right) + \left(1-\alpha\right) + \lambda\left(\beta\right).\end{aligned}$$

The results (3), (4), (5), (6), (7) and (8) can be proved similarly. □

According to Proposition 1.6, the following p-convex and p-concave functions can be constructed.

Example 1.7. [15] *Let $\lambda : (0,\infty) \to \mathbb{R}$, $\lambda(\varkappa) = \varkappa$, then λ is a p-convex function for $p \in (-\infty,0)\cup(0,1]$ and λ is a p-concave function for $p \geq 1$.*

Example 1.8. [15] *Let $\lambda : (0,\infty) \to \mathbb{R}$, $\lambda(\varkappa) = \varkappa^{-p}$, $p \geq 1$, then λ is a p-convex function.*

Example 1.9. [15] *Let* $\lambda : (0, \infty) \to \mathbb{R}$, $\lambda(\varkappa) = -\ln \varkappa$ *and* $p \geq 1$, *then* λ *is a* p*-convex function.*

Example 1.10. [15] *Let* $\lambda : (0, \infty) \to \mathbb{R}$,$\lambda(\varkappa) = \ln \varkappa$ *and* $p \geq 1$, *then* λ *is a* p*-concave function.*

The following Hermite-Hadamard type inequalities were obtained in [15].

Theorem 1.11. [15] *Let* $\lambda : I \subset (0, \infty) \to \mathbb{R}$ *be a* p*-convex function,* $p \in \mathbb{R}\backslash \{0\}$, *and* ϵ, $\varepsilon \in I$ *with* $\epsilon < \varepsilon$. *If* $\lambda \in L[\epsilon, \varepsilon]$, *then we have*

$$\lambda \left(\left[\frac{\epsilon^p + \varepsilon^p}{2} \right]^{\frac{1}{p}} \right) \leq \frac{p}{\varepsilon^p - \epsilon^p} \int_{\epsilon}^{\varepsilon} \frac{\lambda(\varkappa)}{\varkappa^{1-p}} d\varkappa \leq \frac{\lambda(\epsilon) + \lambda(\varepsilon)}{2}. \tag{1. 4}$$

The inequalities (1. 4) are sharp.

Definition 1.12. [19] *Let* $p \in \mathbb{R}\backslash \{0\}$. *A function* $\mu : [\epsilon, \varepsilon] \subset (0, \infty) \to \mathbb{R}$ *is said to be* p*-symmetric with respect to* $\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}}$ *if* $\mu(\varkappa) = \mu\left([\epsilon^p + \varepsilon^p - \varkappa^p]^{\frac{1}{p}}\right)$ *holds for all* $\varkappa \in [\epsilon, \varepsilon]$.

A weighted version of the inequality (1. 4) is proved in [19].

Theorem 1.13. [19] *Let* $\lambda : I \subset (0, \infty) \to \mathbb{R}$ *be a* p*-convex function,* $p \in \mathbb{R}\backslash \{0\}$, ϵ, $\varepsilon \in I$ *with* $\epsilon < \varepsilon$. *If* $\lambda \in L[\epsilon, \varepsilon]$ *and* $w : [\epsilon, \varepsilon] \to \mathbb{R}$ *is non-negative, integrable and* p*-symmetric with respect to* $\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}}$, *then the following inequalities hold*

$$\lambda \left(\left[\frac{\epsilon^p + \varepsilon^p}{2} \right]^{\frac{1}{p}} \right) \int_{\epsilon}^{\varepsilon} \frac{\mu(\varkappa)}{\varkappa^{1-p}} d\varkappa \leq \frac{p}{\varepsilon^p - \epsilon^p} \int_{\epsilon}^{\varepsilon} \frac{\lambda(\varkappa)\mu(\varkappa)}{\varkappa^{1-p}} d\varkappa \tag{1. 5}$$

$$\leq \frac{\lambda(\epsilon) + \lambda(\varepsilon)}{2} \int_{\epsilon}^{\varepsilon} \frac{\mu(\varkappa)}{\varkappa^{1-p}} d\varkappa.$$

For several new Hermite-Hadamard and Fejér type inequalities related to (1. 4) and (1. 5), we refer the interested reader to [15], [19] and [29].

In this article, we prove new integral inequalities of Hermite-Hadamard and Fejér type for differentiable p-convex functions. The results of this paper generalize some known results given in [15] and [29].

2. Main Results

In this section, we recall Gamma, Beta, Hypergeometric functions and some generalizations of the Hölder inequality.

The Gamma function is defined as

$$\Gamma(\varkappa) = \int_0^{\infty} e^{-\alpha} \alpha^{\varkappa - 1} d\alpha.$$

The Beta function, also known as the Euler integral of the first kind, is defined as

$$B(\varkappa, \beta) = \int_0^1 \alpha^{\varkappa - 1} (1 - \alpha)^{\beta - 1} d\alpha, \varkappa > 0, \beta > 0.$$

The hypergeometric function is given as follows

$$ {}_2F_1(\varkappa, \beta; c; z) = \frac{1}{B(\beta, \beta - c)} \int_0^1 \alpha^{\varkappa - 1} (1-\alpha)^{c-\beta-1} (1 - z\alpha)^{-\epsilon} \, d\alpha, $$

where $|z| < 1$ and $c > \beta > 0$.

The weighted Hölder inequality can be stated as follows

$$ \left| \int_{\epsilon}^{\varepsilon} \lambda(\varkappa) \mu(\varkappa) h(\varkappa) \, d\varkappa \right| \leq \left(\int_{\epsilon}^{\varepsilon} |\lambda(\varkappa)|^p h(\varkappa) \, d\varkappa \right)^{\frac{1}{p}} \left(\int_{\epsilon}^{\varepsilon} |\mu(\varkappa)|^q h(\varkappa) \, d\varkappa \right)^{\frac{1}{q}}, $$

where $p, q > 1$ and $p^{-1} + q^{-1} = 1$.

The following result is important to derive the results of this paper.

Lemma 2.1. *Let $\lambda : (c, d) \subset (0, \infty) \to \mathbb{R}$ be a differentiable mapping on (c, d) and $\mu : [\epsilon, \varepsilon] \to [0, \infty)$ be continuous and p-symmetric with respect to $\left(\frac{\epsilon^p + \varepsilon^p}{2}\right)^{\frac{1}{p}}$ for ϵ, $\varepsilon \in (c, d)$ with $\epsilon < \varepsilon$. If $\lambda \in L([\epsilon, \varepsilon])$, then the following equality holds*

$$ \begin{aligned} &\left(\frac{\varepsilon^p - \epsilon^p}{4p} \right) \int_0^1 \left[\int_{L_p(\epsilon, \varepsilon; \alpha)}^{U_p(\epsilon, \varepsilon; \alpha)} \frac{\mu(\varkappa)}{\varkappa^{1-p}} d\varkappa \right] \\ &\quad \times \left[U_{p-1}(\epsilon, \varepsilon; \alpha) \lambda'(U_p(\epsilon, \varepsilon; \alpha)) - L_{p-1}(\epsilon, \varepsilon; \alpha) \lambda'(L_p(\epsilon, \varepsilon; \alpha)) \right] d\alpha \\ &\qquad = \frac{\lambda(\epsilon) + \lambda(\varepsilon)}{2} \int_{\epsilon}^{\varepsilon} \frac{\mu(\varkappa)}{\varkappa^{1-p}} d\varkappa - \int_{\epsilon}^{\varepsilon} \frac{\lambda(\varkappa) \mu(\varkappa)}{\varkappa^{1-p}} d\alpha, \end{aligned} \tag{2. 6} $$

where

$$ U_p(\epsilon, \varepsilon; \alpha) = \left[\left(\frac{1-\alpha}{2} \right) \epsilon^p + \left(\frac{1+\alpha}{2} \right) \varepsilon^p \right]^{\frac{1}{p}} $$

and

$$ L_p(\epsilon, \varepsilon; \alpha) = \left[\left(\frac{1+\alpha}{2} \right) \epsilon^p + \left(\frac{1-\alpha}{2} \right) \varepsilon^p \right]^{\frac{1}{p}}. $$

Proof. Let

$$ I_1 = \left(\frac{\varepsilon^p - \epsilon^p}{4p} \right) \int_0^1 \left[\int_{L_p(\epsilon, \varepsilon; \alpha)}^{U_p(\epsilon, \varepsilon; \alpha)} \frac{\mu(\varkappa)}{\varkappa^{1-p}} d\varkappa \right] U_{p-1}(\epsilon, \varepsilon; \alpha) \lambda'(U_p(\epsilon, \varepsilon; \alpha)) \, d\alpha $$

and

$$ I_2 = \left(\frac{\varepsilon^p - \epsilon^p}{4p} \right) \int_0^1 \left[\int_{L_p(\epsilon, \varepsilon; \alpha)}^{U_p(\epsilon, \varepsilon; \alpha)} \frac{\mu(\varkappa)}{\varkappa^{1-p}} d\varkappa \right] L_{p-1}(\epsilon, \varepsilon; \alpha) \lambda'(L_p(\epsilon, \varepsilon; \alpha)) \, d\alpha. $$

By integration by parts, we have

$$\begin{aligned}
I_1 &= \left(\frac{\varepsilon^p-\epsilon^p}{4p}\right)\int_0^1\left[\int_{L_p(\epsilon,\varepsilon;\alpha)}^{U_p(\epsilon,\varepsilon;\alpha)}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right]U_{p-1}(\epsilon,\varepsilon;\alpha)\,\lambda'(U_p(\epsilon,\varepsilon;\alpha))\,d\alpha\\
&= \frac{1}{2}\int_0^1\left[\int_{L_p(\epsilon,\varepsilon;\alpha)}^{U_p(\epsilon,\varepsilon;\alpha)}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right]d\left(\lambda\left(U_p(\epsilon,\varepsilon;\alpha)\right)\right)d\alpha\\
&= \frac{1}{2}\left[\int_{L_p(\epsilon,\varepsilon;\alpha)}^{U_p(\epsilon,\varepsilon;\alpha)}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right]\lambda(U_p(\epsilon,\varepsilon;\alpha))\Bigg|_0^1\\
&- \left(\frac{\varepsilon^p-\epsilon^p}{2p}\right)\int_0^1\mu(U_p(\epsilon,\varepsilon;\alpha))\,\lambda(U_p(\epsilon,\varepsilon;\alpha))\,d\alpha\\
&= \frac{\lambda(\varepsilon)}{2}\int_\epsilon^\varepsilon\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa-\left(\frac{\varepsilon^p-\epsilon^p}{2p}\right)\int_0^1\mu(U_p(\epsilon,\varepsilon;\alpha))\,\lambda(U_p(\epsilon,\varepsilon;\alpha))\,d\alpha.
\end{aligned}$$

By making the substitution $\varkappa=U_p(\epsilon,\varepsilon;\alpha)$, we get

$$\begin{aligned}
d\alpha &= \left(\frac{2p}{\varepsilon^p-\epsilon^p}\right)\frac{d\varkappa}{U_{p-1}(\epsilon,\varepsilon;\alpha)}\\
&= \left(\frac{2p}{\varepsilon^p-\epsilon^p}\right)\frac{d\varkappa}{(U_p(\epsilon,\varepsilon;\alpha))^{1-p}}=\left(\frac{2p}{\varepsilon^p-\epsilon^p}\right)\frac{d\varkappa}{\varkappa^{1-p}}.
\end{aligned}$$

Hence

$$I_1=\left[\int_\epsilon^\varepsilon\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right]\frac{\lambda(\varepsilon)}{2}-\int_{\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}}}^{\varepsilon}\frac{\lambda(\varkappa)\,\mu(\varkappa)}{\varkappa^{1-p}}d\alpha.$$

Similarly, we can prove that

$$\begin{aligned}
I_2 &= \left(\frac{\varepsilon^p-\epsilon^p}{4p}\right)\int_0^1\left[\int_{L_p(\epsilon,\varepsilon;\alpha)}^{U_p(\epsilon,\varepsilon;\alpha)}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right]L_{p-1}(\epsilon,\varepsilon;\alpha)\,\lambda'(L_p(\epsilon,\varepsilon;\alpha))\,d\alpha\\
&= -\frac{\lambda(\epsilon)}{2}\int_\epsilon^\varepsilon\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa+\int_\epsilon^{\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}}}\frac{\lambda(\varkappa)\,\mu(\varkappa)}{\varkappa^{1-p}}d\alpha.
\end{aligned}$$

This shows that

$$I_1-I_2=\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_\epsilon^\varepsilon\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa-\int_\epsilon^\varepsilon\frac{\lambda(\varkappa)\,\mu(\varkappa)}{\varkappa^{1-p}}d\alpha.$$

This proves the result of the Lemma. □

Remark 2.2. *If $p=1$, the result given in (2. 6) becomes the result proved in* [12, Theorem 2.2].

If $p=-1$, the result of Lemma 2.1 becomes the following important result.

Lemma 2.3. *Let $\lambda:(c,d)\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on (c,d) and $\mu:[\epsilon,\varepsilon]\to[0,\infty)$ be continuous and harmonically-symmetric with respect to $\frac{2\epsilon\varepsilon}{\epsilon+\varepsilon}$ for ϵ,*

$\varepsilon \in (c,d)$ with $\epsilon < \varepsilon$. If $\lambda \in L([\epsilon,\varepsilon])$, then the following equality holds

$$\begin{aligned}\epsilon\varepsilon(\varepsilon-\epsilon)\int_0^1 &\left[\int_{\frac{2\epsilon\varepsilon}{(1+\alpha)\varepsilon+(1-\alpha)\epsilon}}^{\frac{2\epsilon\varepsilon}{(1-\alpha)\varepsilon+(1+\alpha)\epsilon}} \frac{\mu(\varkappa)}{\varkappa^2}d\varkappa\right]\\ &\times\left[\frac{\lambda'\left(\frac{2\epsilon\varepsilon}{(1-\alpha)\varepsilon+(1+\alpha)\epsilon}\right)}{[(1-\alpha)\varepsilon+(1+\alpha)\epsilon]^2}-\frac{\lambda'\left(\frac{2\epsilon\varepsilon}{(1+\alpha)\varepsilon+(1-\alpha)\epsilon}\right)}{[(1+\alpha)\varepsilon+(1-\alpha)\epsilon]^2}\right]d\alpha\\ &=\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_\epsilon^\varepsilon\frac{\mu(\varkappa)}{\varkappa^2}d\varkappa-\int_\epsilon^\varepsilon\frac{\lambda(\varkappa)\mu(\varkappa)}{\varkappa^2}d\alpha. \quad (2.\ 7)\end{aligned}$$

We can now commence to prove the results of this manuscript.

Theorem 2.4. *Let $\lambda : (c,d) \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on (c,d) and $\mu : [\epsilon,\varepsilon] \to [0,\infty)$ be continuous and p-symmetric with respect to $\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}}$ for ϵ, $\varepsilon \in (c,d)$ with $\epsilon < \varepsilon$. If $\lambda \in L([\epsilon,\varepsilon])$ and $\left|\lambda'\right|^q$ is p-convex for $p \in \mathbb{R}\setminus\{0\}$ and $q \geq 1$,*

$$\begin{aligned}&\left|\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_\epsilon^\varepsilon\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa-\int_\epsilon^\varepsilon\frac{\lambda(\varkappa)\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right|\\ &\quad\leq\|\mu\|_\infty\left(\frac{\varepsilon^p-\epsilon^p}{2p}\right)^2\left[\alpha_1(\epsilon,\varepsilon;p,q)\left|\lambda'(\epsilon)\right|^q+\alpha_1(\varepsilon,\epsilon;p,q)\left|\lambda'(\varepsilon)\right|^q\right]^{\frac{1}{q}}, \quad (2.\ 8)\end{aligned}$$

where $\|\mu\|_\infty = \sup_{\varkappa\in[\epsilon,\varepsilon]}|\mu(\varkappa)|$ and

$$\alpha_1(\epsilon,\varepsilon;p,q)=\int_0^1\alpha\left\{\left(\frac{1-\alpha}{2}\right)U_{p-1}^q(\epsilon,\varepsilon;\alpha)+\left(\frac{1+\alpha}{2}\right)L_{p-1}^q(\epsilon,\varepsilon;\alpha)\right\}d\alpha.$$

Proof. Taking the absolute value on both sides of the result of Lemma 2.1 and using the continuous and discrete power-mean inequalities, we have

$$\begin{aligned}&\left|\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_\epsilon^\varepsilon\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa-\int_\epsilon^\varepsilon\frac{\lambda(\varkappa)\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right|\\ &\qquad\leq\left(\frac{\varepsilon^p-\epsilon^p}{4p}\right)\left[\int_0^1\left(\int_{L_p(\epsilon,\varepsilon;\alpha)}^{U_p(\epsilon,\varepsilon;\alpha)}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right)\right]^{1-\frac{1}{q}}\\ &\quad\times\left[\left(\int_0^1\left(\int_{L_p(\epsilon,\varepsilon;\alpha)}^{U_p(\epsilon,\varepsilon;\alpha)}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right)U_{p-1}^q(\epsilon,\varepsilon;\alpha)\left|\lambda'(U_p(\epsilon,\varepsilon;\alpha))\right|^q d\alpha\right)^{\frac{1}{q}}\right.\\ &\quad\left.+\left(\int_0^1\left(\int_{L_p(\epsilon,\varepsilon;\alpha)}^{U_p(\epsilon,\varepsilon;\alpha)}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right)L_{p-1}^q(\epsilon,\varepsilon;\alpha)\left|\lambda'(L_p(\epsilon,\varepsilon;\alpha))\right|^q d\alpha\right)^{\frac{1}{q}}\right]\\ &\leq 2^{1-\frac{1}{q}}\left(\frac{\varepsilon^p-\epsilon^p}{4p}\right)\left[\int_0^1\left(\int_{L_p(\epsilon,\varepsilon;\alpha)}^{U_p(\epsilon,\varepsilon;\alpha)}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right)\right]^{1-\frac{1}{q}}\left\{\int_0^1\left(\int_{L_p(\epsilon,\varepsilon;\alpha)}^{U_p(\epsilon,\varepsilon;\alpha)}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right)\right.\\ &\times\left.\left[U_{p-1}^q(\epsilon,\varepsilon;\alpha)\left|\lambda'(U_p(\epsilon,\varepsilon;\alpha))\right|^q+L_{p-1}^q(\epsilon,\varepsilon;\alpha)\left|\lambda'(L_p(\epsilon,\varepsilon;\alpha))\right|^q\right]d\alpha\right\}^{\frac{1}{q}}. \quad (2.\ 9)\end{aligned}$$

By using the p-convexity of $\left|\lambda^{'}\right|^{q}$ for $q \geq 1$, we get

$$\begin{aligned} & U_{p-1}^{q}(\epsilon,\varepsilon;\alpha)\left|\lambda^{'}\left(U_{p}(\epsilon,\varepsilon;\alpha)\right)\right|^{q}+L_{p-1}^{q}(\epsilon,\varepsilon;\alpha)\left|\lambda^{'}\left(L_{p}(\epsilon,\varepsilon;\alpha)\right)\right|^{q} \\ & \quad \leq\left|\lambda^{'}(\epsilon)\right|^{q}\left\{\left(\frac{1-\alpha}{2}\right)U_{p-1}^{q}(\epsilon,\varepsilon;\alpha)+\left(\frac{1+\alpha}{2}\right)L_{p-1}^{q}(\epsilon,\varepsilon;\alpha)\right\} \\ & \quad +\left|\lambda^{'}(\varepsilon)\right|^{q}\left\{\left(\frac{1+\alpha}{2}\right)U_{p-1}^{q}(\epsilon,\varepsilon;\alpha)+\left(\frac{1-\alpha}{2}\right)L_{p-1}^{q}(\epsilon,\varepsilon;\alpha)\right\}. \end{aligned} \tag{2. 10}$$

By applying (2. 10) in (2. 9), we get

$$\begin{aligned} & \left|\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_{\epsilon}^{\varepsilon}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa-\int_{\epsilon}^{\varepsilon}\frac{\lambda(\varkappa)\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right| \\ & \leq 2^{1-\frac{1}{q}}\left(\frac{\varepsilon^{p}-\epsilon^{p}}{4p}\right)\left[\int_{0}^{1}\left(\int_{L_{p}(\epsilon,\varepsilon;\alpha)}^{U_{p}(\epsilon,\varepsilon;\alpha)}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right)\right]^{1-\frac{1}{q}}\left\{\int_{0}^{1}\left(\int_{L_{p}(\epsilon,\varepsilon;\alpha)}^{U_{p}(\epsilon,\varepsilon;\alpha)}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right)\right. \\ & \quad \times\left[\left|\lambda^{'}(\epsilon)\right|^{q}\left\{\left(\frac{1-\alpha}{2}\right)U_{p-1}^{q}(\epsilon,\varepsilon;\alpha)+\left(\frac{1+\alpha}{2}\right)L_{p-1}^{q}(\epsilon,\varepsilon;\alpha)\right\}\right. \\ & \left.\left.+\left|\lambda^{'}(\varepsilon)\right|^{q}\left\{\left(\frac{1+\alpha}{2}\right)U_{p-1}^{q}(\epsilon,\varepsilon;\alpha)+\left(\frac{1-\alpha}{2}\right)L_{p-1}^{q}(\epsilon,\varepsilon;\alpha)\right\}\right]d\alpha\right\}^{\frac{1}{q}}. \end{aligned} \tag{2. 11}$$

Since

$$\begin{aligned} \int_{L_{p}(\epsilon,\varepsilon;\alpha)}^{U_{p}(\epsilon,\varepsilon;\alpha)}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa & \leq\|\mu\|_{\infty}\int_{L_{p}(\epsilon,\varepsilon;\alpha)}^{U_{p}(\epsilon,\varepsilon;\alpha)}\frac{1}{\varkappa^{1-p}}d\varkappa=\frac{\|\mu\|_{\infty}}{p}\left[U_{p}^{p}(\epsilon,\varepsilon;\alpha)-L_{p}^{p}(\epsilon,\varepsilon;\alpha)\right] \\ & =\|\mu\|_{\infty}\,\alpha\left(\frac{\varepsilon^{p}-\epsilon^{p}}{p}\right), \end{aligned}$$

hence

$$\int_{0}^{1}\left(\int_{L_{p}(\epsilon,\varepsilon;\alpha)}^{U_{p}(\epsilon,\varepsilon;\alpha)}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right)d\alpha=\|\mu\|_{\infty}\left(\frac{\varepsilon^{p}-\epsilon^{p}}{2p}\right).$$

Thus, the inequality (2. 11) becomes the following inequality

$$\begin{aligned} & \left|\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_{\epsilon}^{\varepsilon}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa-\int_{\epsilon}^{\varepsilon}\frac{\lambda(\varkappa)\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right|\leq\|\mu\|_{\infty}\left(\frac{\varepsilon^{p}-\epsilon^{p}}{2p}\right)^{2} \\ & \quad \times\left\{\int_{0}^{1}\alpha\left[\left|\lambda^{'}(\epsilon)\right|^{q}\left\{\left(\frac{1-\alpha}{2}\right)U_{p-1}^{q}(\epsilon,\varepsilon;\alpha)+\left(\frac{1+\alpha}{2}\right)L_{p-1}^{q}(\epsilon,\varepsilon;\alpha)\right\}\right.\right. \\ & \left.\left.+\left|\lambda^{'}(\varepsilon)\right|^{q}\left\{\left(\frac{1+\alpha}{2}\right)U_{p-1}^{q}(\epsilon,\varepsilon;\alpha)+\left(\frac{1-\alpha}{2}\right)L_{p-1}^{q}(\epsilon,\varepsilon;\alpha)\right\}\right]d\alpha\right\}^{\frac{1}{q}}. \end{aligned} \tag{2. 12}$$

The inequality (2. 12) is the desired inequality. □

Remark 2.5. *If $\mu(\varkappa)=\frac{p}{\varepsilon^{p}-\epsilon^{p}}$ for all $\varkappa\in[\epsilon,\varepsilon]$ and $p\in\mathbb{R}\backslash\{0\}$, we can get Hermite-Hadamard type inequalities for p-convex functions from the result of Theorem 2.4.*

The following important results can be deduced from the inequality (2. 8).

Corollary 2.6. *According to the inferences of Theorem 2.4 with $q = 1$, then*

$$\left|\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_{\epsilon}^{\varepsilon}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa-\int_{\epsilon}^{\varepsilon}\frac{\lambda(\varkappa)\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right| \leq \|\mu\|_{\infty}\left(\frac{\varepsilon^{p}-\epsilon^{p}}{2p}\right)^{2}\left[\alpha_{1}(\epsilon,\varepsilon;p,1)\left|\lambda^{'}(\epsilon)\right|+\alpha_{1}(\varepsilon,\epsilon;p,1)\left|\lambda^{'}(\varepsilon)\right|\right], \quad (2.\ 13)$$

where $\|\mu\|_{\infty}=\sup_{\varkappa\in[\epsilon,\varepsilon]}|\mu(\varkappa)|$ *and*

$$\alpha_{1}(\epsilon,\varepsilon;p,1)=\int_{0}^{1}\alpha\left\{\left(\frac{1-\alpha}{2}\right)U_{p-1}(\epsilon,\varepsilon;\alpha)+\left(\frac{1+\alpha}{2}\right)L_{p-1}(\epsilon,\varepsilon;\alpha)\right\}d\alpha.$$

Corollary 2.7. *As far as the reasonings of Theorem 2.4 are justified and $p = 1$, then*

$$\left|\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_{\epsilon}^{\varepsilon}\mu(\varkappa)d\varkappa-\int_{\epsilon}^{\varepsilon}\lambda(\varkappa)\mu(\varkappa)d\varkappa\right| \leq \|\mu\|_{\infty}\left(\frac{\varepsilon-\epsilon}{2}\right)^{2}\left[\frac{\left|\lambda^{'}(\epsilon)\right|^{q}+\left|\lambda^{'}(\varepsilon)\right|^{q}}{2}\right]^{\frac{1}{q}}, \quad (2.\ 14)$$

where $\|\mu\|_{\infty}=\sup_{\varkappa\in[\epsilon,\varepsilon]}|\mu(\varkappa)|$

Proof. The proof follows from the fact that

$$\alpha_{1}(\epsilon,\varepsilon;1,q)=\alpha_{1}(\varepsilon,\epsilon;1,q)=\int_{0}^{1}\alpha\left\{\left(\frac{1-\alpha}{2}\right)+\left(\frac{1+\alpha}{2}\right)\right\}d\alpha=\frac{1}{2}.$$

□

Remark 2.8. *The inequality (2. 14) has been proven in* [12, Theorem 2.4]. *If* $\mu(\varkappa)=\frac{1}{\varepsilon-\epsilon}$ *for all* $\varkappa\in[\epsilon,\varepsilon]$*, the result given in (2. 14) turns out to be the result proved in* [30, Theorem 1].

Corollary 2.9. *Letting $q = 1$ in Corollary 2.7, gives the result for convex functions below*

$$\left|\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_{\epsilon}^{\varepsilon}\mu(\varkappa)d\varkappa-\int_{\epsilon}^{\varepsilon}\lambda(\varkappa)\mu(\varkappa)d\varkappa\right| \leq \|\mu\|_{\infty}\left(\frac{\varepsilon-\epsilon}{2}\right)^{2}\left[\frac{\left|\lambda^{'}(\epsilon)\right|+\left|\lambda^{'}(\varepsilon)\right|}{2}\right], \quad (2.\ 15)$$

where $\|\mu\|_{\infty}=\sup_{\varkappa\in[\epsilon,\varepsilon]}|\mu(\varkappa)|$.

Remark 2.10. *If we take* $\mu(\varkappa)=\frac{1}{\varepsilon-\epsilon}$ *for all* $\varkappa\in[\epsilon,\varepsilon]$ *in (2. 15), we get the result proved in* [5, Theorem 2.3].

Corollary 2.11. *If the assumptions of Theorem 2.4 are met and if* $p=-1$, $q>1$ *and* $q\neq\frac{3}{2}$, *then*

$$\left|\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_{\epsilon}^{\varepsilon}\frac{\mu(\varkappa)}{\varkappa^2}d\varkappa-\int_{\epsilon}^{\varepsilon}\frac{\lambda(\varkappa)\mu(\varkappa)}{\varkappa^2}d\varkappa\right|$$
$$\leq\|\mu\|_{\infty}\left(\frac{\varepsilon-\epsilon}{2\epsilon\varepsilon}\right)^2\left[\alpha_1(\epsilon,\varepsilon;-1,q)\left|\lambda^{'}(\epsilon)\right|^q+\alpha_1(\varepsilon,\epsilon;-1,q)\left|\lambda^{'}(\varepsilon)\right|^q\right]^{\frac{1}{q}},\quad(2.\ 16)$$

where $\|\mu\|_{\infty}=\sup_{\varkappa\in[\epsilon,\varepsilon]}|\mu(\varkappa)|$ *and*

$$\alpha_1(\epsilon,\varepsilon;-1,q)$$
$$=\frac{2^{2q-1}\epsilon^{2q}\varepsilon^{2q}(\epsilon+\varepsilon)^{2-2q}(\varepsilon-5\epsilon+2\epsilon q-2\varepsilon q)+\epsilon^2\varepsilon^{2q}(\epsilon+3\varepsilon-2\varepsilon q+2\epsilon q)}{(\epsilon-\varepsilon)^3(q-1)(2q-1)(2q-3)}$$
$$+\frac{\epsilon^{2q}\varepsilon\left[(2q-1)^2\varepsilon^2+2(q-1)(2q-3)\epsilon^2-\left(4q^2-14q+3\right)\epsilon\varepsilon\right]}{(\epsilon-\varepsilon)^3(q-1)(2q-1)(2q-3)}.$$

Remark 2.12. *If we take* $\mu(\varkappa)=\frac{\epsilon\varepsilon}{\varepsilon-\epsilon}$ *for all* $\varkappa\in[\epsilon,\varepsilon]$ *in (2. 16), we get Hermite-Hadamard type inequalities for harmonically-convex functions.*

Theorem 2.13. *Let* $\lambda:(c,d)\subset(0,\infty)\to\mathbb{R}$ *be a differentiable mapping on* (c,d) *and* $\mu:[\epsilon,\varepsilon]\to[0,\infty)$ *be continuous and p-symmetric with respect to* $\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}}$ *for* ϵ, $\varepsilon\in(c,d)$ *with* $\epsilon<\varepsilon$. *If* $\lambda\in L([\epsilon,\varepsilon])$ *and* $\left|\lambda^{'}\right|^q$ *is p-convex for* $p\in\mathbb{R}\backslash\left\{0,\frac{s}{s-2},\frac{s}{s-1}\right\}$ *and* $s,q>1$, $s\neq2$, *then the following inequality holds*

$$\left|\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_{\epsilon}^{\varepsilon}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa-\int_{\epsilon}^{\varepsilon}\frac{\lambda(\varkappa)\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right|$$
$$\leq\left(\frac{\varepsilon^p-\epsilon^p}{2p}\right)^2\|\mu\|_{\infty}\left\{(\alpha_2(\epsilon,\varepsilon,p;s,x))^{\frac{1}{s}}\left[\frac{\left|\lambda^{'}(\epsilon)\right|^q+5\left|\lambda^{'}(\varepsilon)\right|^q}{12}\right]^{\frac{1}{q}}\right.$$
$$\left.+(\alpha_1(\varepsilon,\epsilon,p;s,-x))^{\frac{1}{s}}\left[\frac{5\left|\lambda^{'}(\epsilon)\right|^q+\left|\lambda^{'}(\varepsilon)\right|^q}{12}\right]^{\frac{1}{q}}\right\},\quad(2.\ 17)$$

where $\|\mu\|_{\infty}=\sup_{\varkappa\in[\epsilon,\varepsilon]}|\mu(\varkappa)|$,

$$\alpha_2(\epsilon,\varepsilon,p;s,x)=\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{s}{p}-s}\left[\frac{p^2-p(1-x)^{\frac{s}{p}-s+1}(p+px+sx-psx)}{(ps-s-2p)(ps-s-p)x^2}\right],$$

$$x=\frac{\epsilon^p-\varepsilon^p}{\epsilon^p+\varepsilon^p}\text{ and }s^{-1}+q^{-1}=1.$$

Proof. From Lemma 2.1 and employing the weighted version of the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_{\epsilon}^{\varepsilon}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa-\int_{\epsilon}^{\varepsilon}\frac{\lambda(\varkappa)\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right| \\
&\leq\left(\frac{\varepsilon^{p}-\epsilon^{p}}{4p}\right)\int_{0}^{1}\left[\int_{L_p(\epsilon,\varepsilon;\alpha)}^{U_p(\epsilon,\varepsilon;\alpha)}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right] \\
&\times\left[U_{p-1}(\epsilon,\varepsilon;\alpha)\left|\lambda'(U_p(\epsilon,\varepsilon;\alpha))\right|+L_{p-1}(\epsilon,\varepsilon;\alpha)\left|\lambda'(L_p(\epsilon,\varepsilon;\alpha))\right|\right]d\alpha \\
&\leq\left(\frac{\varepsilon^{p}-\epsilon^{p}}{2p}\right)^{2}\|\mu\|_{\infty}\int_{0}^{1}\alpha\left[U_{p-1}(\epsilon,\varepsilon;\alpha)\left|\lambda'(U_p(\epsilon,\varepsilon;\alpha))\right|\right. \\
&\left.+L_{p-1}(\epsilon,\varepsilon;\alpha)\left|\lambda'(L_p(\epsilon,\varepsilon;\alpha))\right|\right]d\alpha\leq\left(\frac{\varepsilon^{p}-\epsilon^{p}}{2p}\right)^{2}\|\mu\|_{\infty} \\
&\times\left[\left(\int_{0}^{1}\alpha U_{p-1}^{s}(\epsilon,\varepsilon;\alpha)d\alpha\right)^{\frac{1}{s}}\left(\int_{0}^{1}\alpha\left|\lambda'(U_p(\epsilon,\varepsilon;\alpha))\right|^{q}d\alpha\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{0}^{1}\alpha L_{p-1}^{s}(\epsilon,\varepsilon;\alpha)d\alpha\right)^{\frac{1}{s}}\left(\int_{0}^{1}\alpha\left|\lambda'(L_p(\epsilon,\varepsilon;\alpha))\right|^{q}d\alpha\right)^{\frac{1}{q}}\right]. \quad (2.\ 18)
\end{aligned}$$

Since $\left|\lambda'\right|^{q}$ is p-convex for $p\in\mathbb{R}\backslash\left\{0,\frac{s}{s-2},\frac{s}{s-1}\right\}$ and $s,\,q>1$, $s\neq 2$, we get that

$$\begin{aligned}
\int_{0}^{1}\alpha\left|\lambda'(U_p(\epsilon,\varepsilon;\alpha))\right|^{q}d\alpha&\leq\int_{0}^{1}\alpha\left[\left(\frac{1-\alpha}{2}\right)\left|\lambda'(\epsilon)\right|^{q}+\left(\frac{1+\alpha}{2}\right)\left|\lambda'(\varepsilon)\right|^{q}\right]d\alpha \\
&=\frac{1}{12}\left|\lambda'(\epsilon)\right|^{q}+\frac{5}{12}\left|\lambda'(\varepsilon)\right|^{q} \quad (2.\ 19)
\end{aligned}$$

and

$$\begin{aligned}
\int_{0}^{1}\alpha\left|\lambda'(L_p(\epsilon,\varepsilon;\alpha))\right|^{q}d\alpha&\leq\int_{0}^{1}\alpha\left[\left(\frac{1+\alpha}{2}\right)\left|\lambda'(\epsilon)\right|^{q}+\left(\frac{1-\alpha}{2}\right)\left|\lambda'(\varepsilon)\right|^{q}\right]d\alpha \\
&=\frac{5}{12}\left|\lambda'(\epsilon)\right|^{q}+\frac{1}{12}\left|\lambda'(\varepsilon)\right|^{q}. \quad (2.\ 20)
\end{aligned}$$

Moreover, we also observe that

$$\begin{aligned}
\int_{0}^{1}\alpha U_{p-1}^{s}(\epsilon,\varepsilon;\alpha)d\alpha&=\int_{0}^{1}\alpha\left[\left(\frac{1-\alpha}{2}\right)\epsilon^{p}+\left(\frac{1+\alpha}{2}\right)\varepsilon^{p}\right]^{\frac{s}{p}-s}d\alpha \\
&=\left(\frac{\epsilon^{p}+\varepsilon^{p}}{2}\right)^{\frac{s}{p}-s}\left[\frac{p^{2}-p(1-x)^{\frac{s}{p}-s+1}(p+px+sx-psx)}{(ps-s-2p)(ps-s-p)x^{2}}\right] \quad (2.\ 21)
\end{aligned}$$

and

$$\int_0^1 \alpha L_{p-1}^s(\epsilon,\varepsilon;\alpha)\,d\alpha = \int_0^1 \alpha\left[\left(\frac{1+\alpha}{2}\right)\epsilon^p + \left(\frac{1-\alpha}{2}\right)\varepsilon^p\right]^{\frac{s}{p}-s} d\alpha$$
$$= \left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{s}{p}-s}\left[\frac{p^2 - p(1+x)^{\frac{s}{p}-s+1}(p-px-sx+psx)}{(ps-s-2p)(ps-s-p)x^2}\right], \quad (2.\ 22)$$

where $x = \frac{\epsilon^p-\varepsilon^p}{\epsilon^p+\varepsilon^p}$. The result follows by applying (2. 19)-(2. 22) in (2. 18). □

The following new results for convex and harmonically-convex functions are the direct consequences of Theorem 2.13.

Corollary 2.14. *According to the assumptions of Theorem 2.13 and $p=1$,*

$$\left|\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_\epsilon^\varepsilon \mu(\varkappa)\,d\varkappa - \int_\epsilon^\varepsilon \lambda(\varkappa)\mu(\varkappa)\,d\varkappa\right| \le \left(\frac{\varepsilon-\epsilon}{2}\right)^2\left[\frac{1}{2}\left(\frac{\epsilon-\varepsilon}{\epsilon+\varepsilon}\right)^2\right]^{\frac{1}{s}}\|\mu\|_\infty$$
$$\times\left\{\left[\frac{\left|\lambda'(\epsilon)\right|^q + 5\left|\lambda'(\varepsilon)\right|^q}{12}\right]^{\frac{1}{q}} + \left[\frac{5\left|\lambda'(\epsilon)\right|^q + \left|\lambda'(\varepsilon)\right|^q}{12}\right]^{\frac{1}{q}}\right\}, \quad (2.\ 23)$$

where $\|\mu\|_\infty = \sup_{\varkappa\in[\epsilon,\varepsilon]}|\mu(\varkappa)|$ and $s^{-1}+q^{-1}=1$.

Proof. If $p=1$, we have

$$\alpha_2(\epsilon,\varepsilon,1,s;x) = \alpha_2(\epsilon,\varepsilon,1,s;-x) = \frac{1}{2}\left(\frac{\epsilon-\varepsilon}{\epsilon+\varepsilon}\right)^2.$$

□

Corollary 2.15. *Let the assumptions of Theorem 2.13 be justified and if $p=-1$, then*

$$\left|\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_\epsilon^\varepsilon \frac{\mu(\varkappa)}{\varkappa^2}d\varkappa - \int_\epsilon^\varepsilon \frac{\lambda(\varkappa)\mu(\varkappa)}{\varkappa^2}d\varkappa\right|$$
$$\le \left(\frac{\varepsilon-\epsilon}{2\epsilon\varepsilon}\right)^2\|\mu\|_\infty\left\{(\alpha_2(\epsilon,\varepsilon,-1;q))^{\frac{1}{s}}\left[\frac{\left|\lambda'(\epsilon)\right|^q + 5\left|\lambda'(\varepsilon)\right|^q}{12}\right]^{\frac{1}{q}}\right.$$
$$\left. + (\alpha_2(\varepsilon,\epsilon,-1;q))^{\frac{1}{s}}\left[\frac{5\left|\lambda'(\epsilon)\right|^q + \left|\lambda'(\varepsilon)\right|^q}{12}\right]^{\frac{1}{q}}\right\}, \quad (2.\ 24)$$

where $\|\mu\|_\infty = \sup_{\varkappa\in[\epsilon,\varepsilon]}|\mu(\varkappa)|$, $s^{-1}+q^{-1}=1$ and

$$\alpha_2(\epsilon,\varepsilon,-1;q) = \frac{(\epsilon+\varepsilon)^2(q-1)^2 + 4^{\frac{1}{1-q}}(q-1)\epsilon^{\frac{1+q}{1-q}}(\varepsilon-\epsilon q)(\epsilon+\varepsilon)^{\frac{2q}{q-1}}}{2(\epsilon-\varepsilon)^2(q+1)}.$$

Proof. If $p=-1$, we have

$$\begin{aligned}
&\alpha_2(\epsilon,\varepsilon,-1;s,x)\\
&=\alpha_2(\epsilon,\varepsilon,-1;q)\\
&=\frac{(\epsilon+\varepsilon)^2(q-1)^2+4^{\frac{1}{1-q}}(q-1)\,\epsilon^{\frac{1+q}{1-q}}(\varepsilon-\epsilon q)(\epsilon+\varepsilon)^{\frac{2q}{q-1}}}{2(\epsilon-\varepsilon)^2(q+1)}.
\end{aligned}$$

□

Theorem 2.16. *Let $\lambda:(c,d)\subset(0,\infty)\to\mathbb{R}$ be a differentiable mapping on (c,d) and $\mu:[\epsilon,\varepsilon]\to[0,\infty)$ be continuous and p-symmetric with respect to $\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}}$ for ϵ, $\varepsilon\in(c,d)$ with $\epsilon<\varepsilon$. If $\lambda\in L([\epsilon,\varepsilon])$ and $\left|\lambda'\right|^q$ is p-convex for $p\in\mathbb{R}\backslash\{0,-1\}$ and $s,q>1$, then the following inequality holds*

$$\begin{aligned}
&\left|\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_\epsilon^\varepsilon\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa-\int_\epsilon^\varepsilon\frac{\lambda(\varkappa)\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right|\\
&\quad\le\left(\frac{\varepsilon^p-\epsilon^p}{2p}\right)^2\|\mu\|_\infty\Big\{(\alpha_3(\epsilon,\varepsilon,p,s;x))^{\frac{1}{s}}\Big[\alpha_4(\epsilon,\varepsilon;p;x)\left|\lambda'(\epsilon)\right|^q\\
&\qquad+\alpha_5(\epsilon,\varepsilon;p;x)\left|\lambda'(\varepsilon)\right|^q\Big]^{\frac{1}{q}}+(\alpha_3(\varepsilon,\epsilon,p,s;-x))^{\frac{1}{s}}\\
&\qquad\times\Big[\alpha_5(\varepsilon,\epsilon;p;-x)\left|\lambda'(\epsilon)\right|^q+\alpha_4(\varepsilon,\epsilon;p;-x)\left|\lambda'(\varepsilon)\right|^q\Big]^{\frac{1}{q}}\Big\},\qquad(2.\ 25)
\end{aligned}$$

where $\|\mu\|_\infty=\sup_{\varkappa\in[\epsilon,\varepsilon]}|\mu(\varkappa)|$,

$$\begin{aligned}
\alpha_3(\epsilon,\varepsilon,p,s;x)&=\frac{1}{s+1}\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}-1}{}_2F_1\left(1-\frac{1}{p},s+1,s+2;x\right),\\
\alpha_4(\epsilon,\varepsilon;p;x)&=\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}-1}\frac{p\left[x-p+px+p(1-x)^{\frac{1}{p}+1}\right]}{2(1+p)x^2},\\
\alpha_5(\epsilon,\varepsilon;p;x)&=\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}-1}\left\{\frac{p\left[1-(1-x)^{\frac{1}{p}}\right]}{2x}+\frac{p\left[p-(p+x)(1-x)^{\frac{1}{p}}\right]}{2(1+p)x^2}\right\},
\end{aligned}$$

$$x=\frac{\epsilon^p-\varepsilon^p}{\epsilon^p+\varepsilon^p}\text{ and }s^{-1}+q^{-1}=1.$$

Proof. Applying Lemma 2.1 and using the weighted version of the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_\epsilon^\varepsilon\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa-\int_\epsilon^\varepsilon\frac{\lambda(\varkappa)\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right|\\
&\qquad\le\left(\frac{\varepsilon^p-\epsilon^p}{4p}\right)\int_0^1\left[\int_{L_p(\epsilon,\varepsilon;\alpha)}^{U_p(\epsilon,\varepsilon;\alpha)}\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right]\qquad(2.\ 26)
\end{aligned}$$

$$\times\left[U_{p-1}(\epsilon,\varepsilon;\alpha)\left|\lambda^{'}\left(U_p(\epsilon,\varepsilon;\alpha)\right)\right|+L_{p-1}(\epsilon,\varepsilon;\alpha)\left|\lambda^{'}\left(L_p(\epsilon,\varepsilon;\alpha)\right)\right|\right]d\alpha$$
$$\leq\left(\frac{\varepsilon^p-\epsilon^p}{2p}\right)^2\|\mu\|_\infty\int_0^1\alpha\left[U_{p-1}(\epsilon,\varepsilon;\alpha)\left|\lambda^{'}\left(U_p(\epsilon,\varepsilon;\alpha)\right)\right|\right.$$
$$\left.+L_{p-1}(\epsilon,\varepsilon;\alpha)\left|\lambda^{'}\left(L_p(\epsilon,\varepsilon;\alpha)\right)\right|\right]d\alpha\leq\left(\frac{\varepsilon^p-\epsilon^p}{2p}\right)^2\|\mu\|_\infty$$
$$\times\left[\left(\int_0^1\alpha^sU_{p-1}(\epsilon,\varepsilon;\alpha)\,d\alpha\right)^{\frac{1}{s}}\left(\int_0^1U_{p-1}(\epsilon,\varepsilon;\alpha)\left|\lambda^{'}\left(U_p(\epsilon,\varepsilon;\alpha)\right)\right|^qd\alpha\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\alpha^sL_{p-1}(\epsilon,\varepsilon;\alpha)\,d\alpha\right)^{\frac{1}{s}}\left(\int_0^1L_{p-1}(\epsilon,\varepsilon;\alpha)\left|\lambda^{'}\left(L_p(\epsilon,\varepsilon;\alpha)\right)\right|^qd\alpha\right)^{\frac{1}{q}}\right].\quad(2.\ 27)$$

Since $\left|\lambda^{'}\right|^q$ is p-convex for $p\in\mathbb{R}\setminus\{0,-1\}$ and $q>1$, we get that

$$\int_0^1U_{p-1}(\epsilon,\varepsilon;\alpha)\left|\lambda^{'}\left(U_p(\epsilon,\varepsilon;\alpha)\right)\right|^qd\alpha$$
$$\leq\left|\lambda^{'}(\epsilon)\right|^q\int_0^1\left(\frac{1-\alpha}{2}\right)\left[\left(\frac{1-\alpha}{2}\right)\epsilon^p+\left(\frac{1+\alpha}{2}\right)\varepsilon^p\right]^{\frac{1}{p}-1}d\alpha$$
$$+\left|\lambda^{'}(\varepsilon)\right|^q\int_0^1\left(\frac{1+\alpha}{2}\right)\left[\left(\frac{1-\alpha}{2}\right)\epsilon^p+\left(\frac{1+\alpha}{2}\right)\varepsilon^p\right]^{\frac{1}{p}-1}d\alpha$$
$$=\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}-1}\frac{p\left[x-p+px+p(1-x)^{\frac{1}{p}+1}\right]}{2(1+p)x^2}\left|\lambda^{'}(\epsilon)\right|^q$$
$$+\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}-1}\left\{\frac{p\left[1-(1-x)^{\frac{1}{p}}\right]}{2x}+\frac{p\left[p-(p+x)(1-x)^{\frac{1}{p}}\right]}{2(1+p)x^2}\right\}\left|\lambda^{'}(\varepsilon)\right|^q\quad(2.\ 28)$$

and

$$\int_0^1L_{p-1}(\epsilon,\varepsilon;\alpha)\left|\lambda^{'}\left(L_p(\epsilon,\varepsilon;\alpha)\right)\right|^qd\alpha$$
$$\leq\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}-1}\frac{p\left[x-p+px+p(1-x)^{\frac{1}{p}+1}\right]}{2(1+p)x^2}\left|\lambda^{'}(\varepsilon)\right|^q$$
$$+\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}-1}\left\{\frac{p\left[1-(1-x)^{\frac{1}{p}}\right]}{2x}+\frac{p\left[p-(p+x)(1-x)^{\frac{1}{p}}\right]}{2(1+p)x^2}\right\}\left|\lambda^{'}(\epsilon)\right|^q.\quad(2.\ 29)$$

Moreover, we also observe that

$$\int_0^1 \alpha^s U_{p-1}(\epsilon,\varepsilon;\alpha)\,d\alpha = \int_0^1 \alpha^s \left[\left(\frac{1-\alpha}{2}\right)\epsilon^p + \left(\frac{1+\alpha}{2}\right)\varepsilon^p\right]^{\frac{1}{p}-1} d\alpha$$
$$= \frac{1}{s+1}\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}-1} {}_2F_1\left(1-\frac{1}{p}, s+1, s+2; x\right) \quad (2.\ 30)$$

and

$$\int_0^1 \alpha^s L_{p-1}(\epsilon,\varepsilon;\alpha)\,d\alpha = \int_0^1 \alpha \left[\left(\frac{1+\alpha}{2}\right)\epsilon^p + \left(\frac{1-\alpha}{2}\right)\varepsilon^p\right]^{\frac{1}{p}-1} d\alpha$$
$$= \frac{1}{s+1}\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}-1} {}_2F_1\left(1-\frac{1}{p}, s+1, s+2; -x\right), \quad (2.\ 31)$$

where $x = \frac{\epsilon^p-\varepsilon^p}{\epsilon^p+\varepsilon^p}$. The result follows by applying (2. 28)-(2. 31) in (2. 26). □

From Theorem 2.16 the only result for convex functions can be obtained.

Corollary 2.17. *If the hypotheses of Theorem 2.16 are fulfilled and $p = 1$, then*

$$\left|\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_\epsilon^\varepsilon \mu(\varkappa)\,d\varkappa - \int_\epsilon^\varepsilon \lambda(\varkappa)\mu(\varkappa)\,d\varkappa\right| \le \left(\frac{1}{s+1}\right)^{\frac{1}{s}}$$
$$\times\left(\frac{\varepsilon-\epsilon}{2}\right)^2 \|\mu\|_\infty \left\{\left[\frac{\left|\lambda'(\epsilon)\right|^q + 3\left|\lambda'(\varepsilon)\right|^q}{4}\right]^{\frac{1}{q}} + \left[\frac{3\left|\lambda'(\epsilon)\right|^q + \left|\lambda'(\varepsilon)\right|^q}{4}\right]^{\frac{1}{q}}\right\}, \quad (2.\ 32)$$

where $\|\mu\|_\infty = \sup_{\varkappa\in[\epsilon,\varepsilon]}|\mu(\varkappa)|$ *and* $s^{-1}+q^{-1} = 1$.

Theorem 2.18. *Let $\lambda : (c,d) \subset (0,\infty) \to \mathbb{R}$ be a differentiable mapping on (c,d) and $\mu : [\epsilon,\varepsilon] \to [0,\infty)$ be continuous and p-symmetric with respect to $\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}}$ for ϵ, $\varepsilon \in (c,d)$ with $\epsilon < \varepsilon$. If $\lambda \in L([\epsilon,\varepsilon])$ and $\left|\lambda'\right|^q$ is p-convex for $p \in \mathbb{R}\setminus\left\{-1,-\frac{1}{2},0\right\}$ and $q \ge 1$, then the following inequality holds*

$$\left|\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_\epsilon^\varepsilon \frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa - \int_\epsilon^\varepsilon \frac{\lambda(\varkappa)\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right| \le \left(\frac{\varepsilon^p-\epsilon^p}{2p}\right)^2 \|\mu\|_\infty$$
$$\times\left\{(\beta_1(\epsilon,\varepsilon;p,x))^{1-\frac{1}{q}}\left[\beta_2(\epsilon,\varepsilon;p,x)\left|\lambda'(\epsilon)\right|^q + \beta_3(\epsilon,\varepsilon;p,x)\left|\lambda'(\varepsilon)\right|^q\right]^{\frac{1}{q}}\right.$$
$$\left.(\beta_1(\varepsilon,\epsilon;p,-x))^{1-\frac{1}{q}}\left[\beta_3(\varepsilon,\epsilon;p,-x)\left|\lambda'(\epsilon)\right|^q + \beta_2(\varepsilon,\epsilon;p,-x)\left|\lambda'(\varepsilon)\right|^q\right]^{\frac{1}{q}}\right\}, \quad (2.\ 33)$$

where $\|\mu\|_\infty = \sup_{\varkappa\in[\epsilon,\varepsilon]}|\mu(\varkappa)|$,

$$\beta_1(\epsilon,\varepsilon;p,x) = \left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}-1}\left[\frac{p^2 - p(1-x)^{\frac{1}{p}}(p+x)}{(1+p)x^2}\right],$$

$$\beta_2(\epsilon,\varepsilon;p,x)=\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}-1}\frac{p^2\left[2p(x-1)+x+(1-x)^{1+\frac{1}{p}}(2p+x)\right]}{2(p+1)(2p+1)x^3},$$

$$\beta_3(\epsilon,\varepsilon;p,x)$$
$$=\left(\frac{\epsilon^p+x^p}{2}\right)^{\frac{1}{p}-1}\frac{p\left[p(x+2p(1+x))-(1-x)^{\frac{1}{p}}\left(2x^2+2p^2(1+x)+3px(1+x)\right)\right]}{(p+1)(2p+1)x^3}$$

and

$$x=\frac{\epsilon^p-\varepsilon^p}{\epsilon^p+\varepsilon^p}.$$

Proof. Taking the absolute value on both sides of the result of Lemma 2.1 and using the power-mean inequality, we have

$$\left|\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_\epsilon^\varepsilon\frac{\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa-\int_\epsilon^\varepsilon\frac{\lambda(\varkappa)\mu(\varkappa)}{\varkappa^{1-p}}d\varkappa\right|\leq\left(\frac{\varepsilon^p-\epsilon^p}{2p}\right)^2\|\mu\|_\infty$$
$$\times\left[\left(\int_0^1\alpha U_{p-1}(\epsilon,\varepsilon;\alpha)\,d\alpha\right)^{1-\frac{1}{q}}\left(\int_0^1\alpha U_{p-1}(\epsilon,\varepsilon;\alpha)\left|\lambda'(U_p(\epsilon,\varepsilon;\alpha))\right|^q d\alpha\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1\alpha L_{p-1}(\epsilon,\varepsilon;\alpha)\,d\alpha\right)^{1-\frac{1}{q}}\left(\int_0^1\alpha L_{p-1}(\epsilon,\varepsilon;\alpha)\left|\lambda'(L_p(\epsilon,\varepsilon;\alpha))\right|^q d\alpha\right)^{\frac{1}{q}}\right].\tag{2. 34}$$

Since $\left|\lambda'\right|^q$ is p-convex for $p\in\mathbb{R}\backslash\left\{-1,0,-\frac{1}{2}\right\}$ and $q\geq1$, we get that

$$\int_0^1\alpha U_{p-1}(\epsilon,\varepsilon;\alpha)\left|\lambda'(U_p(\epsilon,\varepsilon;\alpha))\right|^q d\alpha$$
$$\leq\left|\lambda'(\epsilon)\right|^q\int_0^1\alpha\left(\frac{1-\alpha}{2}\right)U_{p-1}(\epsilon,\varepsilon;\alpha)\,d\alpha+\left|\lambda'(\varepsilon)\right|^q\int_0^1\alpha\left(\frac{1+\alpha}{2}\right)U_{p-1}(\epsilon,\varepsilon;\alpha)\,d\alpha$$
$$=\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}-1}\left[\frac{p^2\left[2p(x-1)+x(1-x)^{1+\frac{1}{p}}(2p+x)\right]}{2(p+1)(2p+1)x^3}\left|\lambda'(\epsilon)\right|^q\right.$$
$$\left.+\frac{p\left[p(x+2p(1+x))-(1-x)^{\frac{1}{p}}\left(2x^2+2p^2(1+x)+3px(1+x)\right)\right]}{(p+1)(2p+1)x^3}\left|\lambda'(\varepsilon)\right|^q\right]\tag{2. 35}$$

and

$$\int_0^1 \alpha L_{p-1}(\epsilon,\varepsilon;\alpha)\left|\lambda'(L_p(\epsilon,\varepsilon;\alpha))\right|^q d\alpha$$
$$\le \left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}-1}\left[\frac{p^2\left[2p(x+1)+x(1+\varkappa)^{1+\frac{1}{p}}(2p-x)\right]}{2(p+1)(2p+1)x^3}\left|\lambda'(\varepsilon)\right|^q\right.$$
$$\left.+\frac{p\left[p(x-2p(1-x))+(1+x)^{\frac{1}{p}}\left(2x^2+2p^2(1-x)+3px(1-x)\right)\right]}{2(p+1)(2p+1)x^3}\left|\lambda'(\epsilon)\right|^q\right]. \tag{2. 36}$$

Moreover, we also observe that

$$\int_0^1 \alpha U_{p-1}(\epsilon,\varepsilon;\alpha)\,d\alpha=\int_0^1\alpha\left[\left(\frac{1-\alpha}{2}\right)\epsilon^p+\left(\frac{1+\alpha}{2}\right)\varepsilon^p\right]^{\frac{1}{p}-1}d\alpha$$
$$=\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}-1}\left[\frac{p^2-p(1-x)^{\frac{1}{p}}(p+x)}{(1+p)x^2}\right] \tag{2. 37}$$

and

$$\int_0^1 \alpha L_{p-1}(\epsilon,\varepsilon;\alpha)\,d\alpha=\int_0^1\alpha\left[\left(\frac{1+\alpha}{2}\right)\epsilon^p+\left(\frac{1-\alpha}{2}\right)\varepsilon^p\right]^{\frac{1}{p}-1}d\alpha$$
$$=\left(\frac{\epsilon^p+\varepsilon^p}{2}\right)^{\frac{1}{p}-1}\left[\frac{p^2-p(1+x)^{\frac{1}{p}}(p-x)}{(1+p)x^2}\right], \tag{2. 38}$$

where $\varkappa=\frac{\epsilon^p-\varepsilon^p}{\epsilon^p+\varepsilon^p}$. The result follows by applying (2. 35)-(2. 38) in (2. 34). □

The following interesting Fejér type inequalities for convex functions can be derived from the result of Theorem 2.18.

Corollary 2.19. *If the conditions of Theorem 2.18 are satisfied and if $p=1$, the following Fejér type inequality for convex functions holds*

$$\left|\frac{\lambda(\epsilon)+\lambda(\varepsilon)}{2}\int_\epsilon^\varepsilon\mu(\varkappa)\,d\varkappa-\int_\epsilon^\varepsilon\lambda(\varkappa)\mu(\varkappa)\,d\varkappa\right|\le\left(\frac{\varepsilon-\epsilon}{2}\right)^2\|\mu\|_\infty\left(\frac{1}{2}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\left[\frac{\left|\lambda'(\epsilon)\right|^q+5\left|\lambda'(\varepsilon)\right|^q}{12}\right]^{\frac{1}{q}}+\left[\frac{5\left|\lambda'(\epsilon)\right|^q+\left|\lambda'(\varepsilon)\right|^q}{12}\right]^{\frac{1}{q}}\right\}, \tag{2. 39}$$

where $\|\mu\|_\infty=\sup_{\varkappa\in[\epsilon,\varepsilon]}|\mu(\varkappa)|$.

Remark 2.20. *By choosing $\mu(\varkappa)=\frac{p}{\varepsilon^p-\epsilon^p}$, $\mu(\varkappa)=\frac{1}{\varepsilon-\epsilon}$, $\mu(\varkappa)=\frac{\epsilon\varepsilon}{\varepsilon-\epsilon}$ for all $\varkappa\in[\epsilon,\varepsilon]$, one can get Hermite-Hadamard type inequalities for p-convex functions, convex functions and harmonically-convex functions from Theorem 2.13, Theorem 2.16, Theorem 2.18 and the related corollaries of these theorems.*

3. Comparison of the Results

In this section, we compare the bounds of the results obtained. Let the bounds in Corollary 2.7, Corollary 2.14, Corollary 2.17 and Corollary 2.19 be denoted by $E_1(\epsilon,\varepsilon;q)$, $E_2(\epsilon,\varepsilon;q)$, $E_3(\epsilon,\varepsilon;q)$ and $E_4(\epsilon,\varepsilon;q)$, that is,

$$E_1(\epsilon,\varepsilon;q)=\left[\frac{\left|\lambda'(\epsilon)\right|^q+\left|\lambda'(\varepsilon)\right|^q}{2}\right]^{\frac{1}{q}},$$

$$\begin{aligned}E_2(\epsilon,\varepsilon;q)&=\left[\frac{1}{2}\left(\frac{\epsilon-\varepsilon}{\epsilon+\varepsilon}\right)^2\right]^{1-\frac{1}{q}}\\&\times\left\{\left[\frac{\left|\lambda'(\epsilon)\right|^q+5\left|\lambda'(\varepsilon)\right|^q}{12}\right]^{\frac{1}{q}}+\left[\frac{5\left|\lambda'(\epsilon)\right|^q+\left|\lambda'(\varepsilon)\right|^q}{12}\right]^{\frac{1}{q}}\right\},\end{aligned}$$

$$\begin{aligned}E_3(\epsilon,\varepsilon;q)&=\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\\&\times\left\{\left[\frac{\left|\lambda'(\epsilon)\right|^q+3\left|\lambda'(\varepsilon)\right|^q}{4}\right]^{\frac{1}{q}}+\left[\frac{3\left|\lambda'(\epsilon)\right|^q+\left|\lambda'(\varepsilon)\right|^q}{4}\right]^{\frac{1}{q}}\right\},\end{aligned}$$

and

$$\begin{aligned}E_4(\epsilon,\varepsilon;q)&=\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\\&\times\left\{\left[\frac{\left|\lambda'(\epsilon)\right|^q+5\left|\lambda'(\varepsilon)\right|^q}{12}\right]^{\frac{1}{q}}+\left[\frac{5\left|\lambda'(\epsilon)\right|^q+\left|\lambda'(\varepsilon)\right|^q}{12}\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

We have omitted $\|\mu\|_\infty=\sup_{\varkappa\in[\epsilon,\varepsilon]}|\mu(\varkappa)|$ and $\left(\frac{\varepsilon-\epsilon}{2}\right)^2$ since they are fixed in all these error bounds. Suppose $\lambda(\varkappa)=\frac{\varkappa^{\frac{2}{q}+1}}{\frac{2}{q}+1}$, $\varkappa\in(0,\infty)$, $q>1$, then $\left|\lambda'(\varkappa)\right|^q=\varkappa^2$ is convex. Let us take $\epsilon=1$, $\varepsilon=5$ and $q\in[2,5]$, then it is obvious from Figure 1 that $E_2(\epsilon,\varepsilon;q)$ and $E_4(\epsilon,\varepsilon;q)$ are better error bounds than $E_1(\epsilon,\varepsilon;q)$ and $E_3(\epsilon,\varepsilon;q)$. Indeed, the error bound $E_2(\epsilon,\varepsilon;q)$ is less than all the other error bounds. Hence it reveals that the result of Corollary 2.14 is better than those results given in Corollary 2.7, Corollary 2.17 and Corollary 2.19.

Now we compare the results of Corollary 2.11 and Corollary 2.15. Let the error bounds in Corollary 2.11 and Corollary 2.15 be denoted by $E_5(\epsilon,\varepsilon;q)$ and $E_6(\epsilon,\varepsilon;q)$ respectively. That is,

$$E_5(\epsilon,\varepsilon;q)=\left[\alpha_1(\epsilon,\varepsilon;-1,q)\left|\lambda'(\epsilon)\right|^q+\alpha_1(\varepsilon,\epsilon;-1,q)\left|\lambda'(\varepsilon)\right|^q\right]^{\frac{1}{q}}$$

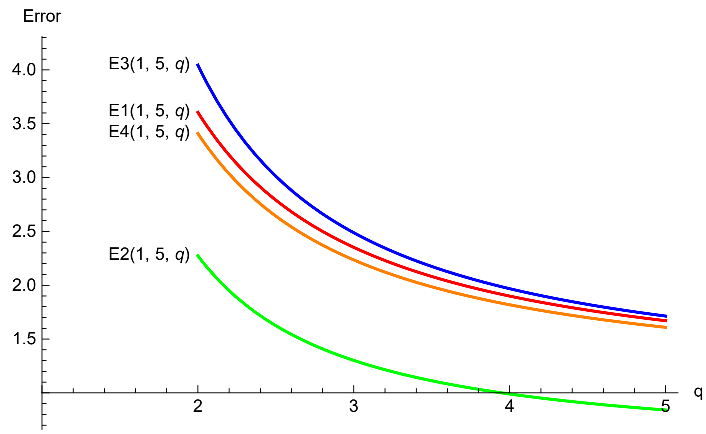

FIGURE 1

and

$$E_5(\epsilon,\varepsilon;q) = (\alpha_2(\epsilon,\varepsilon,-1;s))^{\frac{1}{s}} \left[\frac{\left|\lambda'(\epsilon)\right|^q + 5\left|\lambda'(\varepsilon)\right|^q}{12}\right]^{\frac{1}{q}} + (\alpha_2(\varepsilon,\epsilon,-1;s))^{\frac{1}{s}} \left[\frac{5\left|\lambda'(\epsilon)\right|^q + \left|\lambda'(\varepsilon)\right|^q}{12}\right]^{\frac{1}{q}},$$

where $\alpha_1(\epsilon,\varepsilon;-1,q)$ and $\alpha_2(\epsilon,\varepsilon,-1;q)$ are defined in Corollary 2.11 and Corollary 2.15 respectively. We have omitted the quantity $\left(\frac{\varepsilon-\epsilon}{2\epsilon\varepsilon}\right)^2 \|\mu\|_\infty$ in these error bounds since it is common in them. As we know, $\left|\lambda'(\varkappa)\right|^q = \varkappa^2$, $\varkappa \in (0,\infty)$ is harmonically-convex for $q > 1$. By taking $\epsilon = 1$, $\varepsilon = 5$ and $q \in [2,5]$, it is obvious from Figure 2 that $E_6(\epsilon,\varepsilon;q)$ is a better error bound than $E_5(\epsilon,\varepsilon;q)$.

4. CONCLUSIONS

We have established a new weighted identity involving a differentiable mapping and a non-negative p-symmetric mapping. A number of new integral inequalities of Fejér and Hermite-Hadamard type for differentiable p-convex functions are investigated. A comparison for the different results of the manuscript is demonstrated by drawing graphs using the software Mathematica. We strongly believe that such a comparison of the bounds by using graphs is very useful for the reader as one can compare the results at a glance.

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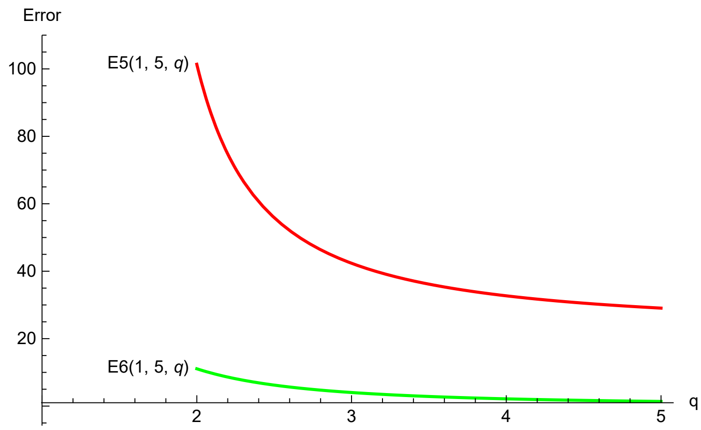

FIGURE 2

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