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Approximate Solutions of Ginzburg-Landau Equation using Two Reliable Techniques

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Abstract. The aim of this article is to compare the Sobolev gradient technique with the Adomian decomposition method for computing a Ginzburg-Landau equation. A convergence criterion for the application of ADM to the generalized Ginzburg-Landau equation is also presented. From the computational point of view, the Sobolev gradient is efficient, easy to use and offer greater accuracy in case of larger domains than ADM.

AMS (MOS) Subject Classification Codes: 35K60; 46C05
Key Words: Sobolev gradients; Steepest descent method, Ginzburg-Landau equation.

1. Introduction

A number of real life problems are modeled as nonlinear differential equations which are not easy to handle due to the difficulties in solving them either analytically or numerically. We encounter such problems in many branches of science and engineering. While solving these problems, we have to make certain assumptions to solve them. The traditional methods usually need linearization, discretization, perturbation or some transformations in order to solve nonlinear systems.

These methods include Adomian decomposition method, introduced by Adomian [2, 3] which provides a quickly convergent series solution and needs no such alterations. Other methods include homotopy decomposition method, Taylor collocation method, differential transform method, homotopy perturbation method, variational iteration method [5, 6, 9, 16, 18, 19, 7, 10, 11, 29] and many more [12, 13, 14, 23, 30, 33, 37]. These methods has success to deal with nonlinear problems, but the region of convergence is not up to the desired solution. To overcome these drawbacks, research has been carried to derive new algorithms based on the gradient techniques.

For the solution of PDEs, it is convenient to construct a functional which represent the sum of square of residues of the equation to be solved and to find the critical point of that functional. The points at which the functional is minimal are the solutions of the given differential equation. This is the basis of steepest descent methods. This method always gives the minimum of the functional. However, the biggest disadvantage of the method is that, if it is used on a badly scaled system to locate the minimum, it will take an infinite number of iterations. The convergence speed is slow since each step taken towards a minimum is very small. If we take larger steps then the result obtained may have large errors. To overcome this problem, the recently developed theory of Sobolev gradients [27] gives a systematic way to deal such problems, both in function spaces and finite dimensional settings. Sobolev gradients have been successfully applied for the solution of ODEs [27, 25] and PDEs [25, 8].

The method has also been used to find minima of energy functionals. Sial et. al. minimized energy functionals related with GL models [34, 35, 36] and their corresponding time evolution is discussed in [31, 32]. Karotson used this method as a preconditioner for nonlinear elliptic problems [22], for the potential equation in electrostatic [20] and semilinear elliptic systems [21]. Knowles used it to solve some inverse problems in groundwater modeling [15] and in elasticity [24]. Nattika and Sauter [28] applied this method successfully to solve differential algebraic equations.

Many researchers have compared the ADM with other existing methods to solve stochastic and deterministic problems. Edwerd et. al. [17] have compared the ADM and RK methods to approximate solutions of predator prey model equations. Wazwaz [38] showed a comparison between ADM and Taylor series methods. In this paper the aim is to compare ADM with some gradient descent methods such as Sobolev gradient method. The solution of the Ginzburg-Landau (GL) equation using ADM has already been provided by Adomian [4] but no convergence analysis of the proposed method is discussed. So, one of our contribution in this article is to furnish that the sufficient condition of convergence for the series solution obtained.

The paper is organized as follow, in section 2, the proposed model is presented, section 3 deals with the ADM applied to G-L equation, in section 4, convergence proof in case of real G-L equation is discussed. Next we apply the Sobolev gradient method on the given model problem. In the subsequent section, numerical results are presented by solving an example and in the end, the summary of the results obtained is discussed.

2. A Model Problem

Consider the GL equation

$$\frac{\partial p}{\partial t} = \gamma p - (k + ic) \mid p \mid^2 p + (\mu + ib)\frac{\partial^2 p}{\partial x^2}, \tag{2. 1}$$

with corresponding initial condition

$$p(x, 0) = g(x), 0 \leq x \leq 1, t > 0$$

where b, c, k, μ and γ are real, $\mu > 0$, $k > 0$. Taking $b, c = 0$, we get the real GL equation

$$\frac{\partial p}{\partial t} = \gamma p - k \mid p \mid^2 p + \mu\frac{\partial^2 p}{\partial x^2}. \tag{2. 2}$$

The aim of this research is to furnish the condition which is sufficient for the convergence of the GL equation in real case.

3. ADOMIAN DECOMPOSITION METHOD (ADM)

To solve Eq. (2. 1) using ADM we define two operators such that

$$L_t \equiv \frac{\partial}{\partial t} \ \ and \ \ L_x \equiv \frac{\partial^2}{\partial x^2}.$$

Writing the given equation in operator form

$$L_t p = (\mu + ib)L_x p - (k + ic) \mid p \mid^2 p + \gamma p. \tag{3. 3}$$

L_t is an invertible operator and L_t^{-1} represents one-fold integration such that, $L_t^{-1}(\cdot) = \int\limits_0^t (\cdot)dt.$

By applying L_t^{-1} on both side of Eq. (3. 3), we have

$$L_t^{-1} L_t p = L_t^{-1}\gamma p - (k + ic)L_t^{-1} \mid p \mid^2 p + (\mu + ib)L_t^{-1} L_x p, \tag{3. 4}$$

$$p(x,t) - p(x,0) = \gamma L_t^{-1} p - (k + ic)L_t^{-1} \mid p \mid^2 p + (\mu + ib)L_t^{-1} L_x p. \tag{3. 5}$$

$$p(x,t) = g(x) + \gamma L_t^{-1} p - (k + ic)L_t^{-1} \mid p \mid^2 p + (\mu + ib)L_t^{-1} L_x p. \tag{3. 6}$$

$p(x,t)$ can be written in decomposition form as

$$p(x,t) = \sum_{n=0}^{\infty} p_n(x,t). \tag{3. 7}$$

Putting Eq. (3. 7) in Eq. (3. 6), we get

$$\begin{aligned} p(x,t) = g(x) + \gamma L_t^{-1}(\sum_{n=0}^{\infty} p_n) - (k + ic)L_t^{-1}(\sum_{n=0}^{\infty} A_n\{\mid p \mid^2 p\}) \\ +(\mu + ib)L_t^{-1} L_x(\sum_{n=0}^{\infty} p_n). \end{aligned} \tag{3. 8}$$

$$p_0 = g(x) = p(x,0), \tag{3. 9}$$

$$p_1 = \gamma L_t^{-1} p_0 - (k + ic)L_t^{-1}(A_0) + (\mu + ib)L_t^{-1} L_x p_0, \tag{3. 10}$$

$$p_2 = \gamma L_t^{-1} p_1 - (k + ic)L_t^{-1}(A_1) + (\mu + ib)L_t^{-1} L_x p_1, \tag{3. 11}$$

$$\vdots \qquad\qquad \vdots$$

$$p_{n+1} = \gamma L_t^{-1} p_n - (k+ic) L_t^{-1}(A_n) + (\mu + ib) L_t^{-1} L_x p_n. \qquad (3.\ 12)$$

where A_n are the Adomian polynomials which need to be evaluated, and an m-term approximant to evaluate p is $\phi_m = \sum_{n=0}^{m} p_n$ that converges to p.
Now to evaluate $A\{| p |^2\}$, we introduce the following functions

$$| p |= p\eta(p); \eta(p) = \mathbf{H}(u) - \mathbf{H}(-u),$$

where **H** and η are the Heaviside step functions of first and second kind respectively. The functions are defined as below

$\mathbf{H} = +1$, for $p > 0$ and 0 for $p < 0$,

and
$\eta = +1$, for $p > 0$ and -1 for $p < 0$.

Therefore,
$| p |^2 \ p = p^3\eta^2(p)$.
Now we find $A\{| p |^2 \ p\}$

$$A_0 = p_0^3,$$

$$A_1 = 3p_0^2 p_1,$$

$$A_2 = 3p_0^2 p_2 + 3p_1^2 p_0,$$

$$A_3 = p_1^3 + 3p_0^2 p_3 + 6p_0 p_1 p_2,$$

$$A_4 = 3p_0^2 p_4 + 3p_1^2 p_2 + 3p_2^2 p_0 + 6p_0 p_1 p_3,$$

$$A_5 = 3p_0^2 p_5 + 3p_1^2 p_3 + 3p_2^2 p_1 + 6p_0 p_1 p_4 + 6p_0 p_2 p_3,$$

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By putting these values into Eq. (3.8), we get all the components of z. The analytical solution can be found by using the approximation

$$p(x,t) = \lim_{n\to\infty} \phi_n. \qquad (3.\ 13)$$

4. Convergence Analysis

Consider the Hilbert space $H = L^2((c,d) \times [0,\tau])$, the set of continuous function from $(c,d) \times [0,\tau]$ to $\mathcal{R}$, define $z : (c,d) \times [0,\tau] \to \mathcal{R}$ with the inner product

$$< p, w >_H = \int_{(c,d)\times[0,\tau]} p(x,s)w(x,s)dsdx, \qquad (4.\ 14)$$

and

$$\int_{(c,d)\times[0,\tau]} p^2(x,s)dsdx < +\infty.$$

The associated norm is defined as

$$|| \ p \ ||_H^2 = < p, p >_H = \int_{(c,d)\times[0,\tau]} p^2(x,s)dsdx. \qquad (4.\ 15)$$

Next we consider the GL equation in operator form

$$L(\phi) = \mu \frac{\partial^2 \phi}{\partial x^2} - k \mid \phi \mid^2 \phi + \gamma\phi. \qquad (4.\ 16)$$

In the following theorem, we prove the convergence of our proposed method.

Theorem 4.1. *The ADM method applied to the GL equation converges towards a solution if the operator L(ϕ) satisfies the following hypotheses:*

$H_1 : \langle L(\phi) - L(\psi), \phi - \psi\rangle \ \geq K(\phi,\psi)\|\phi - \psi\|^2, M(\phi,\psi) > 0, \ \forall \ \phi, \psi \in H.$

H_2 : *For any* $K > 0, \exists C(K) > 0$*, such that for* $\phi, \psi \in H$*, with* $\|\phi\| \leq K, \|\psi\| \leq K$ *, we have* $\langle L(\phi) - L(\psi), y\rangle \leq C(K)\|\phi - \psi\|\|y\|$ *for every* $\psi \in H$*.*

Proof. First, we prove the H_1 (the strong monotonicity) for the operator L(ϕ)

$$\begin{aligned} L(\phi) - L(\psi) &= \mu\frac{\partial^2}{\partial x^2}(\phi - \psi) - k(\eta^2(\phi)\phi^3 - \eta^2(\psi)\psi^3) + \gamma(\phi - \psi) \\ &= -\mu\frac{-\partial^2}{\partial x^2}(\phi - \psi) + k(\eta^2(\phi)[-(\phi - \psi)\sum_{i=0}^{2}\phi^{2-i}\psi^i \\ &+ \gamma(\phi - \psi). \end{aligned} \qquad (4.\ 17)$$

Now we have

$$\langle L(\phi) - L(\psi), \phi - \psi\rangle = -\mu\langle\frac{-\partial^2}{\partial x^2}(\phi - \psi), \phi - \psi\rangle$$

$$+k\eta^2(\phi)\langle-(\phi - \psi)\sum_{i=0}^{2}\phi^{2-i}\psi^i, \phi - \psi\rangle + \gamma\langle\phi - \psi, \phi - \psi\rangle. \qquad (4.\ 18)$$

By using the Cauchy-Schwarz inequality along with the property of $\frac{\partial^2}{\partial x^2}$ in H, we get

$$\begin{aligned}\langle \frac{\partial^2}{\partial x^2}(\phi-\psi), \phi-\psi\rangle &\leq \|\frac{\partial^2}{\partial x^2}(\phi-\psi)\|\|\phi-\psi\| \\ &\leq \delta_1\|\phi-\psi\|^2, \end{aligned} \tag{4. 19}$$

or

$$\langle \frac{-\partial^2}{\partial x^2}(\phi-\psi), \phi-\psi\rangle \geq -\delta_1\|\phi-\psi\|^2. \tag{4. 20}$$

Again by using Cauchy-Schwartz inequality, we have

$$\langle(\phi-\psi)\sum_{i=0}^{2}\phi^{2-i}\psi^i, \phi-\psi\rangle \leq 3K^2\|\phi-\psi\|^2. \tag{4. 21}$$

Hence

$$\langle-(\phi-\psi)\sum_{i=0}^{2}\phi^{2-i}\psi^i, \phi-\psi\rangle \geq -3K^2\|\phi-\psi\|^2. \tag{4. 22}$$

Similarly, we can write

$$\langle\phi-\psi, \phi-\psi\rangle = \|\phi-\psi\|^2. \tag{4. 23}$$

By using Eqs (18), (20) and (22) in Eq. (17), we have

$$\begin{aligned}\langle L(\phi)-L(\psi), \phi-\psi\rangle &\geq \mu\delta_1\|\phi-\psi\|^2 - 3k\eta^2(\phi)M^2\|\phi-\psi\|^2 + \gamma\|\phi-\psi\|^2 \\ &\geq [\mu\delta_1 - 3k\eta^2(\phi)M^2 - \gamma]\|\phi-\psi\|^2\end{aligned}$$

$$\langle L(\phi)-L(\psi), \phi-\psi\rangle \geq \mathbf{M}\|\phi-\psi\|^2,\ \mathbf{M}\geq 0 \tag{4. 24}$$

where $\mathbf{M} = \mu\delta_1 - 3k\eta^2(\phi)K^2 - \gamma$, which implies that $\delta_1 \geq \frac{3k\eta^2(\phi)K^2+\gamma}{\mu}$.

Next we prove the H_2 hypothesis

$$\begin{aligned}\langle L(\phi)-L(\psi), y\rangle = \mu\langle\frac{\partial^2}{\partial x^2}(\phi-\psi, y\rangle - k\eta^2(\phi)\langle(\phi-\psi)\sum_{i=0}^{2}\phi^{2-i}\psi^i, y\rangle \\ +\gamma\langle(\phi-\psi), y\rangle.\end{aligned} \tag{4. 25}$$

$$\begin{aligned}\langle L(\phi)-L(\psi), y\rangle &\leq \mu\delta_2\|\phi-\psi\|\|y\| - k\eta^2(\phi)3M^2\|\phi-\psi\|\|y\| + \gamma\|\phi-\psi\|\|y\| \\ &\leq [\mu\delta_2 + k\eta^2(\phi)3M^2 + \gamma]\|\phi-\psi\|\|y\| \\ &= C(K)\|\phi-\psi\|\|y\|,\end{aligned} \tag{4. 26}$$

where

$$C(K) = \mu\delta_2 + 3kK^2\eta^2(\phi) + \gamma > 0, \tag{4. 27}$$

which completes the second axiom of the proof. □

5. SOBOLEV GRADIENT METHOD

The current section is devoted to define steepest descent in continuous space using Sobolev gradients. A comprehensive analysis associated with frame work of Sobolev gradients is presented in [27].

Let us consider that H denotes a complete inner product space, G is a real valued C^1 function from H to R. By the Riesz representation theorem, for each $x \in H$, there exists a unique member of H, denoted by $\nabla_H G$ such that

$$J'(y)h =< h, \nabla_H J(y) >_H,\ y, h \in H. \tag{5. 28}$$

Define the gradient of J at y to be $\nabla_H J$. Each inner product has an associated gradient and descent direction. To speed up the minimization process, the selection of an inner product space is crucial and vital in steepest descent method.

A number of gradients for a function J can be constructed by choosing different inner product spaces which have diverse numerical chracteristics. If the gradient of a function J is defined in a Sobolev space, then we call that gradient a Sobolev gradient. For the detailed introduction of Sobolev spaces, readers are advised to see [1]. The steepest descent method can be divided in two categories: continuous steepest descent and discrete steepest descent. Let $\nabla_H J$ be as given by Eq. (5. 28) than discrete steepest descent means a process of constructing a sequence $\{y_k\}$ and the starting point of the sequence y_0

$$y_k = y_{k-1} - \delta_k(\nabla_H J)(y_{k-1}),\ k = 1, 2, ... \tag{5. 29}$$

For each iteration k, δ_k is chosen in such a way that it minimizes, if possible,

$$J(x_{k-1} - \delta_k(\nabla_H J)(y_{k-1})). \tag{5. 30}$$

The function $p : [0, \infty) \to H$ in contrast to continuous steepest descent, is constructed such that

$$\frac{dp}{dt} = -\nabla G(p(t)),\ p(0) = p_{initial}. \tag{5. 31}$$

With appropriate conditions on G, $p(t) \to p_\infty$ where $G(p_\infty)$ represents the minima of G. Thus (5. 29) can be interpreted to a numerical simulation procedure for approximating results of (5. 31). Regarding the existence and uniqueness of the continuous steepest descent method, we have the following theorems due to Neuberger:

Theorem 5.1. *Let G be a non-negative $C^{(1)}$ function (a differentiable function whose derivative is continuous) on a Hilbert space H which has a locally lipschizian gradient. Then, for each $x \in H$, there exists a unique function $p : [0, \infty) \to H$ with*

$$p(0) = x,$$

$$p'(t) = -(\nabla_H G)p(t), \quad t \geq 0.$$

Theorem 5.2. *Suppose all the conditions in the hypothesis of the above theorem are satisfied and*

$$\lim_{t\to\infty} p(t) = u,$$

exists, then

$$\nabla_s G(p(t)) = 0.$$

Next, consider the GL Eq. (2. 2) on the interval [0,1] with the given initial conditions. To solve the equation numerically, our instigation is based on gradient descent methods. A number of gradients of functionals is constructed to employ these methods.

Consider a vector $p \in R^M$ on a patterned rectangular mesh. We symbolize by L_2 or H_0^2 the vector space R^M provided with the standard inner product $< p, \eta >= \sum_j p(j)\eta(j)$. The operators $D_0, D_1, D_{11} : R^M \to R^{M-2}$ are defined by

$$D_0(p)(j) \;=\; p(j+1), \tag{5. 32}$$

$$D_1(p)(j) \;=\; \frac{p(j+2) \;-\; p(j)}{2\delta_x}, \tag{5. 33}$$

$$D_{11}(p)(j) \;=\; \frac{(p(j+2) - 2p(j+1) + p(i))}{\delta_x^2}, \tag{5. 34}$$

for $j = 1, 2, .., M-2$ and $\delta_x = \frac{1}{(M-1)}$ denotes the spacing between the nodes. D_0 is the averaging operator, and picks all the points in the grid except endpoints. The operators D_1 and D_{11} estimates the first and second derivatives by using standard central difference formulas. The theoretical development in this paper does not affect by the choice of central difference formula, other possibilities may also be feasible.

The numerical description of the given problem to evolve from one time t to $t + \delta_t$ is to compute

$$D_0\left((1-\gamma\delta_t)p + k\delta_t p^3 - f\right) - \mu\delta_t D_{11}(p) = 0, \tag{5. 35}$$

where f present in the equation is value of p at the previous time and p is the p desired at the next time level. Define $F \in R^{M-2}$ by

$$F(p) \;=\; D_0\left((1-\gamma\delta_t)p + k\delta_t p^3 - f\right) - \mu\delta_t D_{11}(p), \tag{5. 36}$$

which is zero when we have the desired p. Next we define a functional $G(p)$

$$G(p) \;=\; \frac{< F(p), F(p) >}{2}, \tag{5. 37}$$

is zero if $F(p)$ is zero.

5.1. **Gradients and minimization.** The gradient $\nabla G(p) \in R^M$ of a functional $G(p)$ in L_2 is found by solving

$$G(p+h) \;=\; G(p) + < \nabla G(p), h > + O(h^2), \tag{5. 38}$$

for some test functions h. To increase the functional fastest one has to move in the gradient direction. Thus to decrease the functional fastest, one has to move in the opposite direction $-\nabla G(p)$. This is the footing of any steepest descent algorithm. So to reduce $G(p)$, we replace an initial p with $p - \lambda \nabla G(p)$. Here λ is the step size taken towards minimum. For the next iteration, once again we find the gradient and step taken towards minimum. The process is repeated until either $G(p)$ or $\nabla G(p)$ is smaller than some set tolerance. The step size λ can vary for each iteration. To find optimal λ some line minimization routine can be used, but in our experiments we used fixed value of it.
In this particular case,

$$\nabla G(p) \;=\; \left[(1 - \gamma\delta_t + 3k\delta_t p^2) D_0^t F(p) - \mu\delta_t D_{11}^t F(p)\right], \tag{5. 39}$$

gives the coveted gradient.
The steepest descent performed in L_2 space is inefficient. As we increase the dimension of the problem or increase the number of nodes, the steps taken to reach minimum increase substantially. Rather than abandoning steepest descent, we look for some other spaces in which a gradient is defined which overcomes this inefficiency. Neuberger [27] saw that the gradient defined in L_2 space is rough, therefore compelling us to choose small λ. So if the gradient is smooth than one can use bigger λ. He suggested that a better way to find the minimum of a functional is to do the minimization in an appropriate Sobolev space better suited to the problem.
A new space H_2^2 in R^M with the inner product is defined as

$$\begin{aligned}(p,\eta)_s \;=\;\; & < D_0(p), D_0(\eta) > \;+\; < D_1(p), D_1(\eta) > \;+ \\ & < D_{11}(p), D_{11}(\eta) >,\end{aligned} \tag{5. 40}$$

because $F(p)$ and $G(p)$ have D_{11} in them. By following Mahavier's idea to construct a weighted Sobolev space we defined a new Hilbert space $\hat{H}_2^2$ in R^M. The inner product in this new space is

$$\begin{aligned}(p,\eta)_w \;=\;\; & (1-\delta_t)^2 < D_0(p), D_0(\eta) > \;+\; < D_1(p), D_1(\eta) > \;+ \\ & (\delta_t)^2 < D_{11}(p), D_{11}(\eta) >,\end{aligned} \tag{5. 41}$$

because this inner product considers the coefficients of D_{11} and D_0 in $F(p)$ and $G(p)$.
The desired Sobolev gradients $\nabla_s G(p)$, $\nabla_w G(p)$ in H_2^2 and $\hat{H}_2^2$ is found by solving

$$\left(D_0^t D_0 \;+\; D_1^t D_1 \;+\; D_{11}^t D_{11}\right) \nabla_s G(p) \;=\; \nabla G(p), \tag{5. 42}$$

$$\left((1-\delta_t)^2 D_0^t D_0 \;+\; D_1^t D_1 \;+\; (\delta_t)^2 D_{11}^t D_{11}\right) \nabla_w G(p) \;=\; \nabla G(p), \tag{5. 43}$$

respectively.

6. Numerical Results

Consider the problem

$$p_t - p + \mid p \mid^2 p - p_{xx} = x(1 - t + x^2 t^3), \tag{6. 44}$$

with the condition

$$p(x,0) = 0. \tag{6. 45}$$

Now Eq. (16) becomes

$$L_t p - p + \mid p \mid^2 p - L_x p = f(x,t), \tag{6. 46}$$

therefore,

$$p(x,t) = p(x,0) + L_t^{-1} p_n - L_t^{-1} A_n + L_t^{-1} L_x p_n + L_t^{-1}(f(x,t)). \tag{6. 47}$$

The decomposition series solution $p(x,t)$ into $\sum\limits_{n=0}^{\infty} p_n(x,t)$ gives the term by term

$$p_0 = p(x,0) + L_t^{-1}(f(x,t)) = xt - \frac{1}{2}xt^2 + \frac{1}{4}x^3 t^4 \tag{6. 48}$$

$$\begin{aligned} p_1 = L_t^{-1} p_0 - L_t^{-1} A_0 + L_t^{-1} L_x p_0 = \frac{1}{2}xt^2 - \frac{1}{6}xt^3 - \frac{1}{4}x^3 t^4 - \frac{1}{10}x^3 t^5 \\ + \frac{1}{5}x^4 t^5 + ..., \end{aligned} \tag{6. 49}$$

and so on, on the similar pattern we can find the other components as well.
Now we solve given example by the Sobolev gradient method. Let $x \in [0,1]$ with the initial condition $p(x,0) \;=\; 0$. Writing equation in operator form

$$F(p) \;=\; D_0 \left((1-\delta_t)p + \delta_t z^3 - x(1 - t + x^2 t^3) - f\right) - \delta_t D_{11}(p), \tag{6. 50}$$

the gradient of the functional is given by

$$\nabla G(p) \;=\; \left[(1 - \delta_t + 3\delta_t p^2) D_0^t F(p) - \delta_t D_{11}^t F(p)\right]. \tag{6. 51}$$

TABLE 1. Comparison of results in L_2, H_2^2, $\hat{H}_2^2$ with $\delta_t = 0.2$ upto 5 time steps.

λ			iterations			CPUs			M
L_2	H_2^2	$\hat{H}_2^2$	L_2	H_2^2	$\hat{H}_2^2$	L_2	H_2^2	$\hat{H}_2^2$	-
5.0×10^{-7}	3.0	1.7	806.735	393	114	78 86 819	0.789	0.281	51
$---$	3.0	1.7	$--$	385	99	$--$	3.593	0.914	101
$---$	3.0	1.7	$--$	371	101	$--$	20.220	5.592	201
$---$	3.0	1.7	$--$	381	100	$--$	138.383	37.628	401

To study the potency of our algorithm, we assume steepest descent in L_2, H_2^2, $\hat{H}_2^2$. To find the gradients in H_2^2, $\hat{H}_2^2$ we solve Eqs. (34) and (35) by some iterative methods such as conjugate gradient method.

To solve equation numerically, we discretize our domain into M nodes with internodal spacing δ. The initial state was set $f = 0.0$ on all nodes initially. The function p was then evolved. The amended value of p was taken as correct when the infinity norm of $\pi G(p))$ was less than 10^{-7}.

To see the efficiency of the algorithm in different spaces, total number of minimization steps and CPU time were recorded in Table 1.

From Table 1 the results in H_2^2 are far good than the results in L_2 but the best results are in $\hat{H}_2^2$.

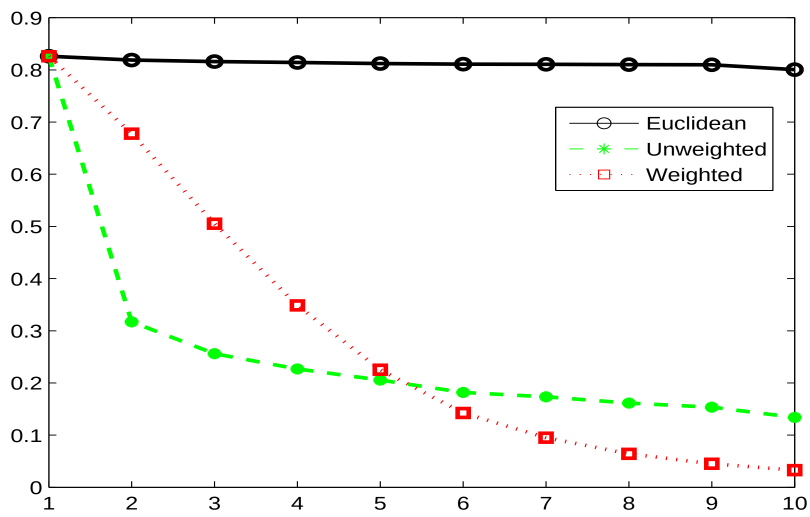

FIGURE 1. Graph of first 10 iterations in comparison with gradients in L_2, H_2^2, $\hat{H}_2^2$.

TABLE 2. Comparison of the approximate solution obtained by ADM and Sobolev gradient methods for time $t = 1$.

x	ADM	$\hat{H}_2^2$
0.0000	0.0000	0.0000
0.0938	0.0934	0.0937
0.2188	0.2118	0.2185
0.3125	0.3117	0.3121
0.4062	0.4039	0.4032
0.5000	0.4823	0.4956
0.6250	0.5963	0.6086
0.7188	0.6814	0.7012
0.8125	0.7709	0.7988
0.9062	0.8523	0.8791
1.0000	0.9543	0.9786
$\parallel$ E $\parallel_\infty$	0.0857	0.0214

In Figure 1 results of using steepest descent in L_2, H_2^2, $\hat{H}_2^2$ for the first ten iterations in comparison with infinity norm of the gradient vector, with an initial value of $u = 0$ is shown. It is quite evident from the graph that the $\hat{H}_2^2$ gradient is the best option for convergence.

To show the comparison with ADM, relative error can be subsequently as

$$\parallel E \parallel_\infty = \max_{i=1,..,M} \left| \frac{p(i) - p_{\text{exact}}}{p_{\text{exact}}} \right|, \tag{6. 52}$$

(where p_{exact} is the exact solution of the given GL equation.

To solve the given example with ADM we take fourth order approximation and we consider the gradient in $\hat{H}_2^2$ space when solving the given problem with descent methods. Table 2 shows the comparative study of Sobolev gradient method with ADM.

From the Table 2, we observe that the presented results are in excellent agreement with ADM.

7. CONCLUSION

In this paper, a comparison is given between the ADM and the Sobolev gradient methods for the solution of GL equation. The obtained results show that the Sobolev gradient method is robust and effective in terms of accuracy. If the domain of the problem becomes larger even than this method still converges. This is not true in the case of ADM, as the rate and region of convergence are potential shortcomings of the method. ADM converges very slowly for wider regions. The truncated solution is also very inaccurate in that region, therefore the application area of the method is very much limited. Also to find the series solution by ADM, the initial state of the system must be known. This is not in the case of Sobolev gradients. By the introduction of suitable weight functions in the construction of Sobolev space its performance can be further improved. Using simple optimization algorithm, for any arbitrary initial guess, this method finds the minimum of a functional.

The choice of underlying space and gradient plays a vital role in the construction of efficient algorithms. A number of different gradients can be defined from the same functional, which have different numerical properties. It is still an open problem as to how we can select a suitable space and define a gradient in it, such that it is best suited for the given problem except for linear problems [26].

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