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Dynamical Behavior of Mathematical Model on the Network of Militants

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Abstract. This work is devoted to the analysis of a mathematical model of the network of militants. To do this, first I introduce the model, then investigate the reproductive number and discuss the stability of model. Next, I obtain the unique strong solution in order to study the dynamical and asymptotic behavior of the model. Finally, I identify conditions based on the parameters of the model, which would be managed by the policy makers, to minimize the number of militants in order to reduce the risk of terrorism.

AMS (MOS) Subject Classification Codes: 00A71; 34D05; 37N40
Key Words: Mathematical model; stability analysis; unique strong solution; dynamical and asymptotic behavior.

1. Introduction

Fundamentalism and terrorism is a severe problem which cause physical as well as mental destruction in human communities. The attack of 11 September 2001 [21], showed that terrorism is capable of inflicting damages 40bn+ and loss of about 3,000 lives that are multiples of the worst U.S. natural disasters. Hurricane Andrew about 20bn+ and 40 to 60 lives [21]. Policy makers always try to make analysis ([7, 10, 13, 20, 34]) in order to reduce this risk.

Fundamentalism is not a disease like Hepatitis B virus, Aids, Polio etc. which spreads through their carriers, exchange of blood etc. Fundamentalism spreads like smoking through close interaction with the groups of militants. A person becomes a militant occasionally for many different causes. Some becomes militant because their family members or friends are militants, some are misguided by the groups of militants and others are attracted etc. Some militants migrate to some areas to achieve their purposes.

Mathematical modeling plays an enormous role to analyze the network of militants and to set the number of police in order to minimize terrorism risk in the effected area. Mathematical models have the potential to educate both policymakers and scholars working in the related fields. In this regards, Major [21] introduced a particular function which gives the probability of plan attack of the militants and their success. Pate et al. [23] presented a probabilistic model on terrorist threats which accounts the probabilities of different scenarios, the objectives of both the U.S. and terrorists, and the dynamic competition between them. Willis et al. [35] presented a probabilistic terrorism risk model and estimated annualized loss from terrorist attacks. They also estimated the critical risk reduction. Chatterjee et al. [8] developed a regional terrorism risk assessment model and discussed the validity, strengths, and limitations of the model in the context of a case study application within the United States. Jeffs et al. [18] studied the mathematics of epidemiology to the analysis of the growth and decline of the memberships of British political parties which has a direct influence on their political effectiveness. For more detail on this discussion, we refer the readers to [11, 14, 18, 25, 33].

In this paper, I have introduced a deterministic model on the spread and control of militants (such kind of models on the spread and control of smoking and different type of viruses can be found in [9, 22, 5, 12, 30, 23, 17, 32, 36, 37]) and obtain the reproductive number. Next, I identify conditions on the parameters of the model which would be managed by the policymakers to minimize the number of militants.

My proposed model consists of four classes: The potential class, occasional class, militants and quit class (i.e., the people who quit militancy). Effects are introduced on the transitions from potential class to occasional class militants and from there to militants and to quit class.

To minimize the risk of terrorism, this work provides useful information to the policy makers about the militants control techniques and the number of police required in the corresponding area.

This article is organized as follows. Section 2 formulate the model. In Section 3, I obtain the reproductive number and study the stability of the proposed model. Section 4 presents the unique strong solution, asymptotic and dynamical behavior of the model and the graphical justification through data. Section 5 studies optimal conditions on the parameters of the model i.e., the conditions which would be managed by the policy makers to reduce the number of militants. Finally, I present conclusion and references.

2. Model Formulation

In this section, I introduce a deterministic model on the network of militants. This model consists of four classes: potential militants, i.e. people who are not militants yet but might become militants in the future, occasional militants class, militants and quit class i.e., people (former militants) who have quit militancy permanently. In the model, effects are introduced on the transitions from potential to occasional class militants and from there to militants and to quit class.

In the start, people become militants occasionally for many different reasons. Some become militants because their family members or friends are militants, some are misguided by the groups of militants and others are attracted. Some militants migrate from one area to another to achieve their purpose.

To controls these groups, policy makers should allocate a number of police in the effected area. It is known that the number of occasional and militants in the effected area is directly proportional to the number of required police there. The value of proportionality constant depends on psychological effects, the capability of training of policemen, quality of weapons and their use etc.

Assume $(N(t))_{t\geq 0}$ denotes the number of required police at time t in the effected area, $(P(t))_{t\geq 0}$-the size of potential class militants, $(O(t))_{t\geq 0}$-the number of occasional militants, $(M(t))_{t\geq 0}$-the militants, while $(Q(t))_{t\geq 0}$-the number of quit ones at this time.

Assume $N(t)$ is proportional to $O(t)$ and $M(t)$, that is,

$$O(t) = k_1 N_1(t) \quad M(t) = k_2 N_2(t), \text{ where } 0 \leq k_2 \leq k_1 < \infty, \ t > 0, \tag{2. 1}$$

with $N(t) = N_1(t) + N_2(t)$. In the above relations, $N_1(t)$ is the number of police to control $O(t)$ while the proportionality constant k_1 is the the potential of a policeman and similar for $N_2(t)$ and k_2. The value of k_1 and k_2 increases by increasing the quality of weapons, training and courage etc. of the policeman. A policeman can control more occasional militants than fundamental militants so $k_1 \geq k_2$. For given values of k_1 and k_2, required number of police at time t can be obtained using these relations. Proportionality between $O(t)$ and $M(t)$ can be obtained through (2. 1) (see also relation (4. 6)).

Dynamics of the above functions obeys the following system of differential equations (for such type of modeling formulation see [4, 7, 5, 18, 31] etc.)

$$\begin{aligned}
\frac{dP(t)}{dt} &= \lambda - aP(t)M(t) - \alpha P(t), \ P(0) \geq 0, \\
\frac{dO(t)}{dt} &= aP(t)M(t) - (\beta + b)O(t), \ O(0) \geq 0, \\
\frac{dM(t)}{dt} &= bpO(t) - (\gamma + c)M(t), \ M(0) \geq 0, \\
\frac{dQ(t)}{dt} &= b(1-p)O(t) + cM(t) - \delta Q(t), \ Q(0) \geq 0,
\end{aligned} \tag{2. 2}$$

where λ is the constant rate at which the potential class increases due to births and migration, α, β, γ and δ are natural death rates of potential, occasional, militants and quit class respectively. a is the transmission coefficient from potential class (the posit value of a increases by increasing the activities of militants), b is transition rate from the occasional class militants in which the flow from occasional class to militants is bpO and to quit ones is $b(1-p)O$, where $p \in [0,1]$. Similarly, c represents the rate at which militants quit militancy. I want to mention that every person moving from $P(t)$ to $M(t)$ or $Q(t)$ passes through a thinking period so stay in $O(t)$ for some time. Moreover, I assume that all the coefficients in the model are locally Lipschitz.

Next section studies the stability analysis of the proposed model.

3. Stability of the Model and the Reproductive Number

In this section, I find militants the free equilibrium point and the unique equilibrium point of the model by using theory of stability analysis.

3.1. **Militants Free Stage.** When there is no militant individual in the community, that is, $O = M = 0$ (for examples see [9, 22, 36, 37]), I obtain the free equilibrium point as $E_0 = \left(\frac{\lambda}{\alpha}, 0, 0, 0\right)$ and the positive equilibrium point, in the term of reproductive number $R_0 = \frac{ab\lambda p}{\alpha(\beta+b)(\gamma+c)}$, as $E_1 = \left(\frac{\lambda}{\alpha R_0}, \frac{\alpha(\gamma+c)}{abp}(R_0 - 1), \frac{\alpha}{a}(R_0 - 1), \frac{\alpha(\gamma+c-\gamma p)}{\delta ap}(R_0 - 1)\right)$.

Next, I discuss the stability of the model at points E_0 and E_1. To do this, the corresponding Jacobian matrix J is given as

$$J = \begin{pmatrix} -aM-\alpha & 0 & -aP & 0 \\ aM & -(\beta+b) & aP & 0 \\ 0 & bp & -(\gamma+c) & 0 \\ 0 & b(1-p) & c & -\delta \end{pmatrix}. \tag{3. 3}$$

Using the free equilibrium point E_0, the Jacobian matrix (3. 3) becomes

$$J_0 = \begin{pmatrix} -\alpha & 0 & -\frac{a\lambda}{\alpha} & 0 \\ 0 & -(\beta+b) & \frac{a\lambda}{\alpha} & 0 \\ 0 & bp & -(\gamma+c) & 0 \\ 0 & b(1-p) & c & -\delta \end{pmatrix}$$

with eigen values

$$\begin{aligned} \lambda_1 &= -\alpha, \\ \lambda_2 &= \frac{-(\beta+b+\gamma+c) + \sqrt{(\beta+b+\gamma+c)^2 - 4\left((\beta+b)(\gamma+c) - \frac{ab\lambda p}{\alpha}\right)}}{2}, \\ \lambda_3 &= \frac{-(\beta+b+\gamma+c) - \sqrt{(\beta+b+\gamma+c)^2 - 4\left((\beta+b)(\gamma+c) - \frac{ab\lambda p}{\alpha}\right)}}{2}, \\ \lambda_4 &= -\delta. \end{aligned}$$

I note that, for every time t, real part of each eigen value is negative if and only if $R_0 < 1$. Hence I come to the following result.

Theorem 3.2. *The model (2. 2) is locally asymptotically stable at point E_0 if and only if $R_0 < 1$.*

Using the positive equilibrium point E_1, the Jacobian matrix J takes the form:

$$J_1 = \begin{pmatrix} -\alpha R_0 & 0 & -\alpha(R_0-1) & 0 \\ \alpha(R_0-1) & -(\beta+b) & \alpha(R_0-1) & 0 \\ 0 & bp & -(\gamma+c) & 0 \\ 0 & b(1-p) & c & -\delta \end{pmatrix}.$$

Using Matlab software, it is found that real part of all the eigen values of the latter matrix is negative if $R_0 > 1$. With this, I come to the following result:

Theorem 3.3. *The system (2. 2) is locally asymptotically stable at E_1 if and only if $R_0 > 1$.*

4. Solution and dynamical behavior of the model

This section studies the existence of the non-negative unique strong solution of the model and its analysis.

The assumption that the coefficients in each equation of the model are locally Lipschitz leads that, for non-negative initial values $P(0)$, $O(0)$, $M(0)$ and $Q(0)$, each equation has unique strong solution [4, 29].

Next, I show the positivity of the solution of the model.

The explicit solution of the first equation in model (2. 2) is given by

$$P(t) = P(0)e^{-\int_0^t (aM(u)+\alpha)du} + \lambda \int_0^t e^{-\int_s^t (aM(u)+\alpha)du} ds, \quad t \geq 0. \tag{4. 4}$$

Solution to the second equation in the model can be expressed as

$$O(t) = e^{-(\beta+b)t}\left[O(0) + a\int_0^t e^{(\beta+b)s} P(u)M(u)du\right], \ t \geq 0, \tag{4. 5}$$

explicit solution of the third equation is

$$M(t) = e^{-(\gamma+c)t}\left[M(0) + bp\int_0^t e^{(\gamma+c)s} O(s)ds\right], \ t \geq 0, \tag{4. 6}$$

while, the fourth equation gives

$$Q(t) = e^{-\delta t}\left[Q(0) + \int_0^t e^{\delta s}(b(1-p)O(s) + cM(s))ds\right], \ t \geq 0. \tag{4. 7}$$

Next, using data collected from valley Swat khyber pakhtunkhwa Pakistan, where violence of militants started in the beginning of 2009. This data is collected in two different timings: one is the time where militants were too active and facing no resistance from any side i.e., the duration of the time from beginning of 2009 up to the mid of 2009 and another is the time when Pakistan army launched the operation ”Operation Zarb-e-Azb” against them.

Month-wise data before the launch of operation (see for detail [1, 2, 15, 16, 26, 3, 24, 19, 27, 6, 28]) is given as:

Symbol	Value
a	0.00001576507
b	0.20635
c	0.0022
λ	0.015
α	0.008
β	0.008
γ	0.008
δ	0.008
p	0.7

where $P(0) = 439140$, $O(0) = 9000$, $M(0) = 1300$ and $Q(0) = 560$.

Using expressions (4. 4)-(4. 7) and values given in the latter table, one have

$$\frac{dP(t)}{dt} \leq 0, \ \frac{dO(t)}{dt} \geq 0, \ \frac{dM(t)}{dt} \geq 0 \text{ and } \frac{dQ(t)}{dt} \geq 0,$$

which show that the function $P(t)$ decreases while $O(t)$, $M(t)$ and $Q(t)$ increase with time. Using this and the condition

$$P(t) + O(t) + M(t) + Q(t) = \textit{ the total population},$$

one concludes that $O(t) + M(t) + Q(t)$ converges to the total population.

Dynamical behavior of these functions is shown in the following figure.

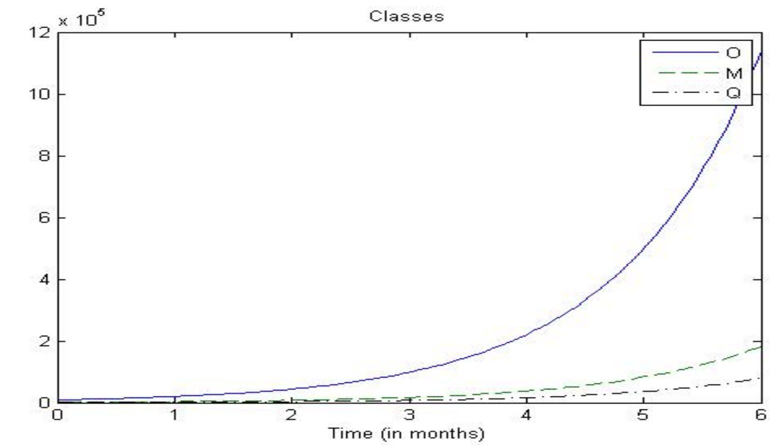

FIGURE 1. Figure shows the dynamical and asymptotic behaviour of the model before the launch of operation Zarb-e-Azb.

Using the references [1, 2, 15, 16, 26, 3, 24, 19, 27, 6, 28], month-wise data after the launch of operation is given as:

Symbol	Value
a	0.00001572327
b	350
c	0.003
λ	0.0001
α	0.008
β	0.01
γ	0.43
δ	0.008
p	0.08

where $P(0) = 318000$, $O(0) = 100000$, $M(0) = 20000$ and $Q(0) = 10000$.

I want to mentioned that the value of a decreased due to restrictions on militants.

Dynamical and Asymptotic behavior of $O(t)$, $M(t)$ and $O(t)$ is shown in the following figure.

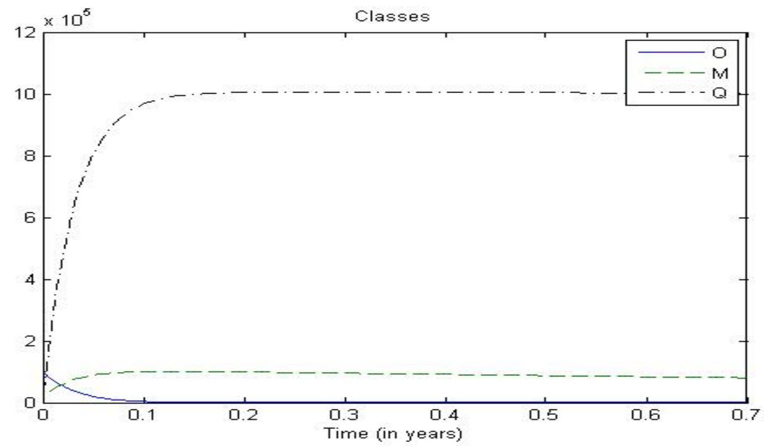

FIGURE 2. This figure shows the dynamic and asymptotic behaviour of the model after the launch of operation.

5. Optimal condition

In this section, I identify conditions based on the parameters of the model which would be managed to minimize the number of militants.

From the first data one can show that the parameters of the model satisfy the conditions

$$\lambda > \alpha + a,\ \gamma + c + aP(0) > \beta + b,\ \delta + aP(0) > \beta + b,\ \delta < \gamma + c, \tag{5. 8}$$

while the second data satisfies

$$\lambda < \alpha + a,\ \gamma + c + aP(0) < \beta + b,\ \delta + aP(0) < \beta + b,\ \delta < \gamma + c. \tag{5. 9}$$

By checking all the equality and inequality relations between λ and $\alpha + a$, $\gamma + c + aP(0)$ and $\beta + b$, $\delta + aP(0)$ and $\beta + b$ and δ and $\gamma + c$, I found that the best controlling strategy is the strategy under which the parameters of the model satisfy

$$\lambda < \alpha + a,\ \gamma + c + aP(0) < \beta + b,\ \delta + aP(0) < \beta + b,\ \delta > \gamma + c. \tag{5. 10}$$

Under the conditions (5. 10), one find that $\frac{dM(t)}{dt} < 0$ as the time t tends to infinity. That is, $M(t)$ decreases. Hence any controlling strategy under which the latter condition is satisfied, will control the militancy.

The graphical behavior of condition (5. 10) is given as

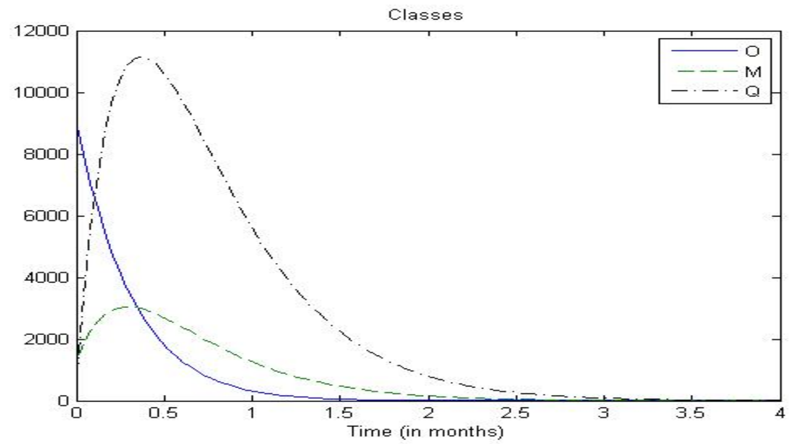

FIGURE 3. This figure shows the dynamic and asymptotic behaviour of $O(t)$, $M(t)$ and $O(t)$ under the latter conditions.

Note: For given values of k_1 and k_2 and every fixed time t, the policy makers can use (2. 1) to calculate the required number $N(t)$ of police. Using this, they can decide to hire or drop the surplus number of police.

6. Conclusion

In this work, I have introduced a mathematical model on the evolution of the number of different level of militants. Results show that managing some parameters of the model, the number of militants and required number of police can be reduced to a minimum number.

Financial and non-financial competing interests

I have neither patents nor applying for patents relating to the contents of this manuscript. I have no non-financial competing interests (political, personal, religious, ideological, intellectual, commercial or any other) which are related to this manuscript.

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