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Existence and Uniqueness of Solutions of Mathematical Models of Predator Prey Interactions

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Abstract. In the study of ecological sciences predator-prey models are very beneficial and they are frequently used because dynamics of animal populations can easily be observed and researchers can also predict that how they will develop over with the passage of time. The objective of this paper is to verify the existence and uniqueness of solutions of mathematical models of predator prey interactions determined in Shakil et al. [19, 20] through a quasi-chemical approach in order to study the predator prey populations and discover the tendencies visible. Each case started with a set of initial and boundary conditions that formed different consequences for the functions of the populations of predator (foxes) and prey (rabbits).

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1. INTRODUCTION

1.1. **Existence, Uniqueness, and Geometry of Solutions.** The Lotka Voltera models are a couple of first order nonlinear differential equations which were originally introduced by Alfred J. Lotka [14] and Vito Volterra [21], these equations are also famous as the predator prey models.

If F is the number of predator (foxes) population and R is the number of prey (rabbits) population, then this model will have a conventional mathematical representation as

$$\begin{aligned} \frac{dF}{dt} &= AFR - BF, \\ \frac{dR}{dt} &= CR - DFR. \end{aligned} \tag{1. 1}$$

The derivatives $\frac{dF}{dt}$ and $\frac{dR}{dt}$ represent the growth of two populations with respect to time,A, B, C and D are parameters describing the two species interaction in the following way:

- A-represents how many foxes can live off of the rabbits per year.
- B-represents how many die off naturally per year.
- C- represents how quickly rabbits reproduce.
- D- is how many rabbits get eaten by foxes per year.

Differential equations have been a fundamental tool for modeling the natural world. At the opening moment of the universe if we are familiar accurately with the laws of nature and with the situation of the universe, the situation at a subsequent of the same universe can easily be predicted. These equations have provided key insights into catastrophic shifts in ecosystems, dynamics of disease outbreaks, mechanisms maintaining biodiversity, and stabilizing forces in food webs.

For these equations, one might be interested in understanding how the species' density changes in time and how these temporal changes depend on its initial density. Often, ecological systems involve many moving parts with multiple types of interacting individuals in which case describing their dynamics involves systems of ordinary differential equations. A classic example of this type is the Lotka-Volterra predator-prey equations that describe the dynamics between a prey and its predator.

After writing down a differential equation model of an ecological system, a modeler wants to know how the variables of interest change in time. For instance, how are densities of the prey and predator changing relative to one another? Moreover, how does this dynamic depend on the initial densities of both populations? To answer these types of questions, one needs to find solutions to the differential equation.

If a solution to the initial/boundary value problem exists, one has to wonder whether it is unique.

After all, if it is not unique, one may never know if all solutions have been uncovered and which, if any, are biologically relevant. Biologically plausible things can still happen. Even if solutions to the initial/boundary value problems exist and are unique, they might not be defined for all time.

2. HISTORICAL BACKGROUND OF WORK ON EXISTENCE AND UNIQUENESS OF PREDATOR PREY MODELS

Here in this article we provide criteria for the existence and uniqueness of solutions for the predator prey mathematical models of two species under certain initial and boundary conditions. The prime objective of the article is to extend and sharpen prevailing consequences on the existence and uniqueness of solutions to predator prey two species models. We provide conditions for existence and uniqueness and we give improved bounds for two species models hence these improved bounds will sharpen the known results.

The Lotka-Volterra model is criticized due to the hypothesis of the infinite growth of the prey population in the absence of a predator and as being impractical generally for its structural instability. However, it is a constructive tool having the essential properties of the actual predator-prey systems, and assists as a robust basis from which more sophisticated models can be developed, Murray [16].

A great deal of work is done on the predator prey competition models, particularly by taking the case study of two species. A few of the illustrative articles are [l-15, 17, 22]. About the work prior to 1986, a more detailed bibliography is debated in [1] by Ali. The articles, by Blate and Brown [2], Dancer [8], Li and Logan [13] provide a qualitative treatment under dirichlet boundary conditions for two species model. A very detailed and comprehensive analysis is presented in Dancer [8] but it still needs to answer some open questions. For three species models a few related complications are discussed in Lakos [12]. Many species case of the existence of a coexistence states under Neumann boundary conditions are treated in the articles, Ali [1], Brown [3,4], Mckeznna [15], Korman and Leung [10]. But theoretically the case of Neumann boundary conditions is easier comparatively because Neumann boundary conditions are always satisfied by the solutions of the equivalent system of ordinary differential equations and in the reaction-diffusion system they are used for the purpose of assessment. For two competing species case a reasonably simple and quantitative conditions are provided in the articles [5-7, 11, 15, 17, 22] for the coefficients which involve stability, existence and uniqueness of positive steady states. For two species model a few results yielding uniqueness are given in, Cantrell and Cosner [6], Cixner [7], Mckeznna [15], Korman and Leung [11]. Further, we refer the reader for the existence and uniqueness of solutions for problems to [27-28] There is a qualitative difference between the case of two species and the case of many species model, because with two species model it is possible to make the system quasimonotone but with three or more species it will be not possible. Thus, quasimonotonicity methods cannot be applied to larger systems. The articles, Blate and Brown [2], Cixner [7], Dancer [8], Cantrell and Cosnero [5] debate the cases in which the Lotka-Volterra model retains numerous coexistence states. The stability analysis of predator-prey population model with time delay and constant rate of harvesting can be seen in [24]. Further, an Alternative approach to the persistence in a 3 species predator-prey modal is given in [25]. The survival curves for the model of growth and decay of tumor cab be seen in [26].

For coexistence states the question of uniqueness is moderately delicate. In this article conditions for stability and uniqueness of solutions of the coexistence state for the case of two species are given and some quantitative outcomes are improved. An estimate is obtained here for a quantity befalling in the assumptions of some of the uniqueness results. Our assessment improves some of the existing results in the studies of ecological systems and partially negative answers are given to the conjectures of many researchers. These result yields an algebraically computable criterion for the positive coexistence of competing species of animals in many biological models.

3. MAIN IDEAS

3.1. **A Simple Cell Jump Model, The Origin Of Our Work.**

Let us consider that our space is divided into two cells, a system represented as a chain of cells each with homogeneous composition and elementary transfer process (for us there are only two cells) between them. Let us numerate these cells by the Roman numbers *I* and *II* and mark all the components and quantities related to them by the upper index *I* or *II*, correspondingly (Fig. 1).

On each cell we have some concentrations for these processes. Let c^I is the vector of concentration in the first cell *I* and c^{II} is the vector of concentration in the second cell *II* .

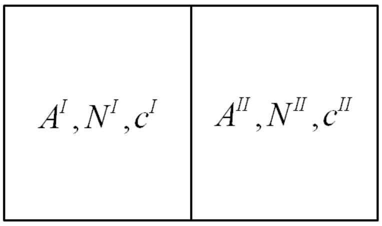

FIGURE 1. Cell Jump Model

Where c^I, c^{II}represent the concentrations of the components of A^I, A^{II} respectively and N^I, N^{II} represent the vectors of composition of the respective cells.

3.2. **Simple Diffusion.** The simplest mechanism of diffusion between any two cells is the process of jumping of particles from one cell to another neighbouring cell. This type of mutually inverted and mutually inverse process can be written as, Gorban et al. [9],

$$\begin{aligned} &A^I \to A^{II}, \\ &A^{II} \to A^I. \end{aligned} \tag{3. 2}$$

Suppose that w_r^I and w_r^{II} represent the reaction rates in the respective cells and in the continuous limit we get the Fick law Gorban et al. [9] as the first Taylor approximation. To get the continuous limit, we take $c^I = c\left(x\right)$,$c^{II} = c\left(x+l\right)$ and use the Taylor expansion: $c\left(x+l\right) = c\left(x\right) + l\partial_x c + o\left(l^2\right)$. Then by using the mass action law, total flux J_F will be calculated as

$$J_F = \frac{kl\left[c^{II} - c^I\right]}{l} = D\nabla c. \tag{3. 3}$$

In this approximation, $kl = D$, where l is the cell size.

4. A QUASI CHEMICAL BASED APPROACH FOR PREDATOR-PREY INTERACTIONS

Let us suppose that our space is divided on two territories, on each territory we have some concentrations (number of foxes and rabbits) for these processes. Let F^I represents (fox/ foxes in the first territory) and R^Irepresents (rabbit/ rabbits in first territory) is the vector concentration in territory *I* and F^{II} represents (fox/ foxes in second territory) and R^{II} represents (rabbit/ rabbits in second territory) is the vector of concentration in the second territory *II*. The territories for the foxes are considered to be the simple cells. The interactions between predator (fox/foxes) and its prey (rabbit/rabbits) are represented by the chemical reactions which obey the mass action law. Here we specially refer this modeling by Gorban et al. [9] who developed the idea and modeled diffusion equations for different mechanisms and proved the dissipation inequalities.

The cell-jump models may be considered as the proper diffusion models by themselves, the territorial animal like fox is given a simple cell as its territory. The sense in which the discrete equations for cells converge to the partial differential equations of diffusion is that the cell models give the semi-discrete approximation of the partial differential equations for diffusion which result in a system of ordinary differential equations in cells. There are jumps of concentrations on the boundary of cells. The system of semi discrete models for cells with “no-flux” boundary conditions has all the nice properties of the chemical kinetic equations for closed systems. Under the proper relations between coefficients, like complex balance or detailed balance, this system demonstrated globally stable dynamics.

4.1. **Mechanism Of Circulation.** Between two foxes/rabbits the mechanism of circulation is the procedure when a fox/ rabbit move from that territory which is defined for it, to another territory which is defined for another fox/rabbit and vise versa. The foxes are territorial animals; they define proper areas for them to stay. This mechanism is illustrated as

$$\begin{aligned} &F^I \rightarrow F^{II}, \\ &F^{II} \rightarrow F^I. \end{aligned} \tag{4. 4}$$

Let us suppose that c^I/c^{II}be the concentrations of foxes in the first /second cell and d^I/d^{II}be the concentrations of rabbits in the first/second cells respectively. Then by using the law of mass action, mathematical model for this mechanism is determined by Shakil et al. [19] as

$$\frac{\partial c}{\partial t} = D\Delta c. \tag{4. 5}$$

With initial and boundary conditions as

$$c\left(x,0\right) = c_0\left(x\right), c\left(0,t\right) = 0, c\left(l,t\right) = 0. \tag{4. 6}$$

In this approximation $kl = D$, where k is a constant, l is the cell size and Δ is Laplace operator.

Our above model is a special case of Rahman [18] and existence and uniqueness of solution for this model is already proved by Rahman [18].

4.2. **Mechanism Of Sharing Place.** Let us suppose that a fox F_i^I is present in territory one and another fox F_j^{II} is present in territory two, if fox F_i^I leaves territory one and moves to territory two and vise versa. This mechanism will be described through the equations of the form

$$\begin{aligned} F_i^I + F_j^{II} \rightarrow F_j^I + F_i^{II}, \\ F_i^{II} + F_j^I \rightarrow F_i^I + F_j^{II}. \end{aligned} \tag{4. 7}$$

Mathematical models for this mechanism are determined by Shakil et al. [19] as

$$\frac{\partial c_i}{\partial t} = kl\left[c_j\Delta c_i - c_i\Delta c_j\right], \tag{4. 8}$$

$$\frac{\partial c_j}{\partial t} = kl\left[c_i\Delta c_j - c_j\Delta c_i\right], \tag{4. 9}$$

with initial and boundary conditions as

$$\begin{aligned} c_i\left(x,0\right) = c_0\left(x\right), \quad c_i\left(0,t\right) = 0, \quad c_i\left(l,t\right) = 0, \\ c_j\left(x,0\right) = c_1\left(x\right), \quad c_j\left(0,t\right) = 0, \quad c_j\left(l,t\right) = 0. \end{aligned} \tag{4. 10}$$

For the existence and uniqueness of solution of the above models, we move as;

Adding equations (4. 8) and (4. 9), we get

$$\begin{aligned} \frac{\partial c_i}{\partial t} + \frac{\partial c_j}{\partial t} &= 0 \\ \Rightarrow \qquad \frac{\partial}{\partial t}\left(c_i + c_j\right) &= 0. \end{aligned} \tag{4. 11}$$

Let us suppose that $(c_i + c_j) = u$, then we get

$$\frac{\partial u}{\partial t} = 0, \tag{4. 12}$$

and initial and boundary conditions will reduce to

$$u\left(x, 0\right) = u_0\left(x\right), \qquad u\left(0, t\right) = 0, \qquad u\left(l, t\right) = 0. \tag{4. 13}$$

Proposition 1. If u *is the solution of system* (4. 12) *and* (4. 13) *then the concentrations of* u *satisfies* $\underset{0 \le t \le T}{sup} \left\|u\right\|_{L^2}^2 = \left\|u_0\right\|_{L^2}^2$.

Theorem 1. *Assume* $u_o \in H^2\left(\Omega\right)$ *then the system* (4. 12) *and* (4. 13) *has a unique classical solution* $u(r, t)$*on* $(0, T)$ *where* $H^2\left(\Omega\right)$ *represents Hilbert Space and* $\Omega = [0, \ell]$.

In order to prove the theorem, first we need to prove the proposition.

Proof. Taking inner product of equation (4. 12) with (u) and integrating, we have

$$\begin{aligned} & \int_\Omega u \frac{\partial u}{\partial t} dx = 0 \\ \Rightarrow \quad & \frac{d}{dt} \int_\Omega \left|u\right|^2 dx = 0 \\ \Rightarrow \quad & \frac{d}{dt} \left\|u\right\|_{L^2}^2 = 0 \\ \Rightarrow \quad & \underset{0 \le t \le T}{sup} \left\|u\right\|_{L^2}^2 = \left\|u_0\right\|_{L^2}^2 . \end{aligned} \tag{4. 14}$$

This proves the existence of solution of the above model.

For uniqueness of solution, let us suppose that u_1 and u_2 be two solutions of the equation (4. 12). Therefore, $\frac{\partial u_1}{\partial t} = \frac{\partial u_2}{\partial t} = 0$, and this implies $\frac{\partial}{\partial t}\left(u_1 - u_2\right) = 0$.

Let us suppose that $w = \left(u_1 - u_2\right)$ and $w\left(x, 0\right) = 0$, then $\frac{\partial w}{\partial t} = 0$.

Taking inner product with w and integrating, we have

$$\begin{aligned} & \int_\Omega w \frac{\partial w}{\partial t} dx = 0 \\ \Rightarrow \quad & \frac{d}{dt} \left\|w\right\|_{L^2}^2 = 0 \\ \Rightarrow \quad & \left\|w\right\|_{L^2}^2 = 0 \\ \Rightarrow \quad & \left\|u_1 - u_2\right\|_{L^2}^2 = 0 \\ \Rightarrow \quad & u_1 - u_2 = 0 \\ \Rightarrow \quad & u_1 = u_2. \end{aligned} \tag{4. 15}$$

This proves the uniqueness of solution of the above model.
Again,

$$\begin{aligned} & \|u\|_{L^2}^2 = \|c_i + c_j\|_{L^2}^2 = \|u_0\|_{L^2}^2 \\ \Rightarrow \quad & \|c_i\|_{L^2}^2 - \|c_j\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 \\ \Rightarrow \quad & \|c_i\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 \\ \Rightarrow \quad & \|c_j\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 . \end{aligned} \tag{4. 16}$$

Now by taking foxes and rabbit's case together, this mechanism will be described through the equations of the form

$$F_i^I + R_j^{II} \rightarrow F_i^I + R_j^I \rightarrow F_i^I, \tag{4. 17}$$

$$F_i^{II} + R_j^I \rightarrow F_i^{II} + R_j^{II} \rightarrow F_i^{II}. \tag{4. 18}$$

The above stated mechanism (4. 17) illustrates that a fox is present in territory one and if a rabbit from territory two moves to territory one, fox present in that territory will prey that rabbit and a similar case for territory two is illustrated in mechanism (4. 18).

Mathematical models for this mechanism are determined by Shakil et al. [19] as

$$\frac{\partial c}{\partial t} = kl\left[d\Delta c - c\Delta d + l\Delta d\Delta c\right], \tag{4. 19}$$

$$\frac{\partial d}{\partial t} = kl\left[c\Delta d - d\Delta c + l\Delta c\Delta d\right], \tag{4. 20}$$

with initial and boundary conditions as

$$\begin{aligned} c(x,0) = c_0(x), \quad c(0,t) = 0, \quad c(l,t) = 0, \\ d(x,0) = d_0(x), \quad d(0,t) = 0, \quad d(l,t) = 0. \end{aligned} \tag{4. 21}$$

To find the existence and uniqueness of solution of the above models, we add the equations (4. 19) and (4. 20), to get

$$\begin{aligned} \frac{\partial c}{\partial t} + \frac{\partial d}{\partial t} &= 2kl^2\Delta c\Delta d \\ \Rightarrow \quad \frac{\partial}{\partial t}(c+d) &= 2kl^2\Delta c\Delta d. \end{aligned} \tag{4. 22}$$

Let us suppose that $c + d = v$, then we get

$$\frac{\partial v}{\partial t} = 2kl^2\Delta c\Delta d. \tag{4. 23}$$

The initial and boundary conditions will reduce to

$$v(x,0) = v_0(x), \quad v(0,t) = 0, \quad v(l,t) = 0. \tag{4. 24}$$

Proposition 2. If *v is the solution of system* (4. 23) *and* (4. 24) *then the concentrations of v satisfies* $\sup_{0\leq t\leq T} \|v\|_{L^2}^2 = \|v_0\|_{L^2}^2$.

Theorem 2. *Assume $v_0 \in H^2(\Omega)$ then the system* (4. 23) *and* (4. 24) *has a unique classical solution $v(r,t)$ on $(0,T)$ where $H^2(\Omega)$ represents Hilbert Space and $\Omega = [0,\ell]$.*

Proof. Taking inner product of equation (4. 23) with $(-\Delta v)$ and using integration by parts, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int(\Delta v)^2\,dx &= -2kl^2\int\Delta v\Delta c\Delta d dx\\ &= -2kl^2\int(\nabla v)\,\nabla(\Delta c\Delta d)dx. \qquad (4.\ 25)\end{aligned}$$

Now,

$$\begin{aligned} & 4\Delta c\Delta d=(\Delta c+\Delta d)^2-(\Delta c-\Delta d)^2\\ \Rightarrow\quad & 4\Delta c\Delta d\leq(\Delta v)^2+(\Delta c+\Delta d)^2\\ \Rightarrow\quad & 4\Delta c\Delta d\leq 2\,(\Delta v)^2\\ \Rightarrow\quad & 2\Delta c\Delta d\leq(\Delta v)^2\,. \qquad (4.\ 26)\end{aligned}$$

Then equation (4. 25) implies that

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla v\right\|_{L^2}^2\leq kl^2\int(\nabla v)\,\nabla\,(\Delta v)^2\,dx. \qquad (4.\ 27)$$

Let us suppose that

$$\begin{aligned} & \nabla\left(\nabla v\,(\Delta v)^2\right)=\Delta v\,(\Delta v)^2+\nabla v\nabla\,(\nabla v)^2\\ \Rightarrow\quad & \nabla\left(\nabla v\,(\Delta v)^2\right)-(\Delta v)^3=\nabla v\nabla\,(\nabla v)^2\,. \qquad (4.\ 28)\end{aligned}$$

Then equation (4. 27) implies that

$$\begin{aligned} & \frac{d}{dt}\left\|\nabla v\right\|_{L^2}^2\leq 2kl^2\int\nabla\left(\nabla v\,(\Delta v)^2\right)dx-\int(\Delta v)^3\,dx\\ \Rightarrow\quad & \frac{d}{dt}\left\|\nabla v\right\|_{L^2}^2\leq 0-\int(\Delta v)^3\,dx\\ \Rightarrow\quad & \frac{d}{dt}\left\|\nabla v\right\|_{L^2}^2+\int(\Delta v)^3\,dx\leq 0\\ \Rightarrow\quad & \frac{d}{dt}\left\|\nabla v\right\|_{L^2}^2\leq 0\\ \Rightarrow\quad & \left\|\nabla v\right\|_{L^2}^2\leq\left\|\nabla v_0\right\|_{L^2}^2\,. \qquad (4.\ 29)\end{aligned}$$

Using Poincare inequality, we have

$$\left\|v\right\|_{L^2}^2\leq\left\|\nabla v_0\right\|_{L^2}^2\,.$$

Since $v=c+d$, then,

$$\begin{aligned} & \left\|c\right\|_{L^2}^2-\left\|d\right\|_{L^2}^2\leq\left\|c+d\right\|_{L^2}^2\\ \Rightarrow\quad & \left\|c\right\|_{L^2}^2-\left\|d\right\|_{L^2}^2<\left\|\nabla v_0\right\|_{L^2}^2\\ \Rightarrow\quad & \left\|c\right\|_{L^2}^2\leq\left\|\nabla v_0\right\|_{L^2}^2\,. \qquad (4.\ 30)\end{aligned}$$

Also this implies $\left\|d\right\|_{L^2}^2\leq\left\|\nabla v_0\right\|_{L^2}^2$. Thus existence of solution of the above model follows.

For uniqueness of solution, let us suppose that v_1 and v_2 be two solutions of equation (4. 23) therefore,

$$\frac{\partial v_1}{\partial t} \leq kl^2 \left(\Delta v_1\right)^2, \\ \frac{\partial v_2}{\partial t} \leq kl^2 \left(\Delta v_2\right)^2,$$

which gives

$$\frac{\partial}{\partial t}\left(v_1 - v_2\right) \leq kl^2 \left[\left(\Delta v_1\right)^2 - \left(\Delta v_2\right)^2\right]. \tag{4. 31}$$

For the case, $\left(\Delta v_1\right)^2 \leq \left(\Delta v_2\right)^2$, we have, $\frac{\partial}{\partial t}\left(v_1 - v_2\right) \leq 0$.

Let $\Re = v_1 - v_2$, then $\dfrac{\partial \Re}{\partial t} \leq 0$. Now taking the inner product with $\Re$ and integrating, we have

$$\begin{aligned} & && \int_\Omega \Re \frac{\partial \Re}{\partial t} dx \leq 0 \\ & \Rightarrow && \frac{d}{dt}\left\|\Re\right\|_{L^2}^2 \leq 0 \\ & \Rightarrow && \left\|\Re\right\|_{L^2}^2 \leq 0 \\ & \Rightarrow && \left\|v_1 - v_2\right\|_{L^2}^2 \leq 0 \\ & \Rightarrow && v_1 - v_2 \leq 0 \\ & \Rightarrow && v_1 = v_2. \end{aligned} \tag{4. 32}$$

This proves the uniqueness of solution.

4.3. **Mechanism Of Attraction.** The mechanisms of attraction in the case study of foxes are unusual but in winter seasons they pair up. If two foxes are present in a same territory and a fox from another territory inters in that territory, then it is given by the following equations:

$$\begin{aligned} 2F^I + F^{II} &\rightarrow 3F^I, \\ F^I + 2F^{II} &\rightarrow 3F^{II}. \end{aligned} \tag{4. 33}$$

Mathematical model for this mechanism is determined by Shakil et al. [19] as

$$\frac{\partial c}{\partial t} = kl\frac{1}{3}\Delta c^3. \tag{4. 34}$$

With initial and boundary conditions as

$$c\left(x,0\right) = c_0\left(x\right), \qquad c\left(0,t\right) = 0, \qquad c\left(l,t\right) = 0. \tag{4. 35}$$

Proposition 3. If *c is the solution of system* (4. 34) *and* (4. 35) *then the concentrations of csatisfies* $\sup\limits_{0\leq t\leq T} \left\|c\right\|_{L^2}^2 = \left\|c_0\right\|_{L^2}^2$.

Theorem 3. *Assume* $c_0 \in H^2\left(\Omega\right)$ *then the system* (4. 34) *and* (4. 35) *has a unique classical solution* $c(r,t)$ *on* $(0,T)$ *where* $H^2\left(\Omega\right)$ *represents Hilbert Space and* $\Omega = [0,\ell]$.

Proof. For proving the existence and uniqueness of solutions for these models taking inner product of equation (4. 34) by ∇c and integrating by parts, we have

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\left\|c\right\|_{L^2}^2 = -\frac{kl}{3}\int \nabla c \Delta c^3 dx \\ \Rightarrow \quad & \frac{d}{dt}\left\|c\right\|_{L^2}^2 + \frac{2}{3}kl\int \nabla c \Delta c^3 dx = 0 \\ \Rightarrow \quad & \frac{d}{dt}\left\|c\right\|_{L^2}^2 \leq 0 \\ \Rightarrow \quad & \underset{0\leq t\leq T}{sup}\left\|c\right\|_{L^2}^2 \leq \left\|c_0\right\|_{L^2}^2 . \end{aligned} \tag{4. 36}$$

This proves the existence of solution for the modeled equation of mechanism of attraction and uniqueness of solution of the above model follows from previous models.

4.4. **Mechanism Of Repulsion.** In winter seasons if two foxes are already present in a same territory and a fox from another territory inters that territory then the foxes present in that territory will have a fight with that fox, as a result of which this fox will leave that territory and returns to his initial position then this mechanism will be described by the stoichiometric equations as

$$\begin{aligned} 3F^I &\rightarrow 2F^I + F^{II}, \\ 3F^{II} &\rightarrow F^I + 2F^{II}. \end{aligned} \tag{4. 37}$$

Mathematical model for this mechanism is determined by Shakil et al. [19] as

$$\frac{\partial c}{\partial t} = kl\Delta c^3. \tag{4. 38}$$

The initial and boundary conditions are

$$c\left(x,0\right) = c_0\left(x\right), \quad c\left(0,t\right) = 0, \quad c\left(l,t\right) = 0. \tag{4. 39}$$

This model is a special case of mechanism of attraction and existence and uniqueness of solution for this model follows from the modeled equation of mechanism of attraction.

4.5. **Autocatalysis Mechanism.** Self reproduction process is called autocatalysis process. In our case study, in Shakil et al. [20] we studied that how foxes/ rabbits interact together in an autocatalysis way to enhance their populations.

When in winter months two foxes couple together in territory one/two, we suppose that a male fox $F_{(m)}$ and a female fox $F_{(f)}$ interact together, as a result of that interaction n number of foxes are born, this mechanism will be described as

$$\begin{aligned} 2F^I &\rightarrow n_1 F^I, \\ 2F^{II} &\rightarrow n_2 F^{II}, \end{aligned} \tag{4. 40}$$

where $n_1, n_2 > 2$. Mathematical model for this mechanism is determined by Shakil et al. [20] as

$$\frac{\partial c}{\partial t} = 2klc\Delta c, \tag{4. 41}$$

with initial and boundary conditions as

$$c\left(x,0\right) = c_0\left(x\right), \quad c\left(0,t\right) = 0, \quad c\left(l,t\right) = 0. \tag{4. 42}$$

Proposition 4. If c *is the solution of system* (4. 41) *and* (4. 42) *then the concentrations of* c *satisfies* $\sup\limits_{0\le t\le T} \|c\|^2_{L^2} = \|c_0\|^2_{L^2}$.

Theorem 4. *Assume* $c_0 \in H^2(\Omega)$ *then the system* (4. 41) *and* (4. 42) *has a unique classical solution* $c(r,t)$ *on* $(0,T)$ *where* $H^2(\Omega)$ *represents Hilbert Space and* $\Omega = [0,\ell]$.

Proof. Taking inner product of equation (4. 41) with c and integrating, we have

$$\int_\Omega c\frac{\partial c}{\partial t}dx = 2kl\int c^2\Delta c$$

$$\Rightarrow \quad \frac{d}{dt}\|c\|^2_{L^2} = -2kl\int \nabla c(\nabla c^2)dx$$

$$\Rightarrow \quad \frac{d}{dt}\|c\|^2_{L^2} + 2kl\int \nabla c(\nabla c^2)dx = 0$$

$$\Rightarrow \quad \frac{d}{dt}\|c\|^2_{L^2} \le 0 \tag{4. 43}$$

$$\Rightarrow \quad \|c\|^2_{L^2} - \|c_0\|^2_{L^2} \le 0$$

$$\Rightarrow \quad \|c\|^2_{L^2} \le \|c_0\|^2_{L^2} . \tag{4. 44}$$

Thus, existence of solution for the above mechanism follows and uniqueness of solution for the above model also follows from previous models.

Remarks: Some of our mathematical models determined by Shakil et al. [19, 20] can be treated as special cases of the above discussed models, so existence and uniqueness of solutions for those mathematical models follows from the above models.

5. CONCLUSIONS

The consequences and analysis of this article under the suppositions made demonstrate the existence and uniqueness of solutions of simple predator-prey models and some fascinating biological consequences have been derived and justified. Each case started with a set of initial and boundary conditions that formed different significances for the functions of the populations of predator prey interactions. Subsequently, in more detailed models it will be useful to observe the robustness of these results.

This approach merely delivers a short time results which lead to the unfavorable conclusion. A similar approach can easily be extended to systems with several prey species and single predator, or several predator species and single prey, and an equivalent conclusion will hold. These results lead to a better understanding and give a new insight for the movements of different mechanisms of predator prey interactions.

The consequences of the paper exhibit that, under the raised suppositions, a species may be abolished (efficiently in finite interval of time). The consequences of the study lead to the conclusion that in the process of evolution, predation, competition is not the only reason for destruction of certain species. Our conclusions lead to the results that, not only in significance of ineffective competition, but even when competition is inattentive, the original species are certainly driven to destruction. For example, in the aforementioned foxes and rabbits model, the rabbits may be excluded entirely by predation alone in a finite time.

The consequences of the paper propose and justify our new approach towards the study of ecological problems of predator prey types of interactions. Usually biological study requires the introduction of predator prey populations increasing or decreasing to an adequate level. This predator-prey system after a few time will reach to its coexisting equilibrium, so only a short term results are achieved through this approach. Our results state that this approach will be further effective if, instantaneously with introducing the predators, a prey species is introduced in order to provide a longstanding food source. For the introduced prey it is not essential to contest with the pest for resources. The resources can be a theme of fortification, so if they do not compete it will be much better for them, but the prey species introduced certainly have a better survival capability. Under the circumstances, in finite time the actual pest species will be entirely annihilated.

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