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On G. Bennett's Inequality

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Abstract. The aim of this paper is to establish similar results to that of G. Bennett[2] and C. P. Niculescu[4] in the context of functions which are 3-convex/concave at a point.

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1. Introduction

A vast theory developed in the study of convex functions, may be readily applied to most significant topics in real analysis and economics. In past recent years, a rapid development has experienced in the theory of convex functions. This can be accredited to several causes, two of which are as follows: First, so many areas in modern analysis directly or indirectly involves application of convex functions. Second, convex functions have huge impact on the theory of inequalities and several important inequalities are out-turn of the application of convex functions (see [6]).
In [2], G. Bennett presented some consequences of an inequality describing the behavior of convex functions with respect to a mass distribution. Later, C. P. Niculescu proved an abstract version of this result([4]), which is shown in the next theorem.

Theorem 1.1. [4] *Let $\mathcal{I}$ is an interval carrying a positive Borel measure ℓ and $\mathcal{A}$; $\mathcal{B}$; $\mathcal{C}$ are three disjoint compact subintervals of $\mathcal{I}$ of positive measure. Then*

$$\ell(\mathcal{B}) = \ell(\mathcal{A}) + \ell(\mathcal{C}) \tag{1. 1}$$

and

$$\int_{\mathcal{B}} \alpha d\ell(\alpha) = \int_{\mathcal{A}} \alpha d\ell(\alpha) + \int_{\mathcal{C}} \alpha d\ell(\alpha); \tag{1. 2}$$

give a necessary and sufficient condition under which the inequality

$$\int_{\mathcal{B}} f(\alpha) d\ell(\alpha) \leq \int_{\mathcal{A}} f(\alpha) d\ell(\alpha) + \int_{\mathcal{C}} f(\alpha) d\ell(\alpha). \tag{1. 3}$$

is valid for every convex function $f : \mathcal{I} \to \mathbb{R}$.

Definition 1.2. *(see* [5]*, p. 179). Any real Borel measure ℓ on an interval $\mathcal{I}$ such that $\ell(\mathcal{I}) > 0$ and*

$$\int_{\mathcal{I}} f(\alpha)d\ell(\alpha) \geq 0 \;\; for\ every\ nonnegative\ convex\ function\ f : \mathcal{I} \to \mathbb{R}.$$

is called a Steffensen-Popoviciu measure.

A brief description of this concept is given in the following result, independently due to T. Popoviciu [6] and A. M. Fink [3]:

Lemma 1.3. *Suppose that ℓ be a real Borel measure on an interval $\mathcal{I}$ with $\ell(\mathcal{I}) > 0$. Then ℓ is a Steffensen-Popoviciu measure iff the following condition of endpoints positivity,*

$$\int_{\mathcal{I}\cap(-\infty,t]} (t-\alpha)d\ell(\alpha) \geq 0 \;\; and \;\; \int_{\mathcal{I}\cap[t,\infty)} (\alpha - t)d\ell(\alpha) \geq 0$$

holds for every $t \in [a,b]$.

Theorem 1.4. *(see* [5]*, p. 184-185, for details) Let ℓ is a Steffensen- Popoviciu measure on an interval $\mathcal{I}$. Then the inequality*

$$f(b_\ell) \leq \frac{1}{\ell(\mathcal{I})} \int_{\mathcal{I}} f(\alpha)d\ell(\alpha)$$

holds for every continuous convex function f on $\mathcal{I}$, here $b_\ell = \frac{1}{\ell(\mathcal{I})} \int_{\mathcal{I}} \alpha d\ell(\alpha)$ represents the barycenter of ℓ.

Definition 1.5. [4] *Any real Borel measure ℓ on an interval $\mathcal{I}$ such that $\ell(\mathcal{I}) > 0$ and*

$$\int_{\mathcal{I}} f(\alpha)d\ell(\alpha) \geq 0 \;\; for\ every\ nonnegative\ concave\ function\ f : \mathcal{I} \to \mathbb{R}.$$

is called a dual Steffensen-Popoviciu measure.

Theorem 1.6. *Theorem 1.1 even works if ℓ is a real Borel measure on $\mathcal{I}$ and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are three disjoint subintervals of $\mathcal{I}$ such that the restriction of ℓ to each of the intervals $\mathcal{A}$ and $\mathcal{C}$ is a Steffensen-Popoviciu measure and the restriction of ℓ to $\mathcal{B}$ is a dual Steffensen-Popoviciu measure.*

Now, we consider the inequality of G. Bennett for new class of functions in following section.

2. Main Results

In [1], I. A. Baloch, J. Pecaric, M. Praljak defined a new class of functions which is defined as follow:

Definition 2.1. *Let $\mathcal{I}$ be a non-degenerate interval in $\mathbb{R}$ and c an interior point of it. A function $f : \mathcal{I} \to \mathbb{R}$ is called 3-convex function at point c (respectively 3-concave function at point c) if there exists a constant K such that the function $F(\alpha) = f(\alpha) - \frac{K}{2}\alpha^2$ is concave (resp. convex) on $\mathcal{I} \cap (-\infty, c]$ and convex (resp. concave) on $\mathcal{I} \cap [c, \infty)$.*

A property that explains the name of the class is the fact that a function is 3-convex on an interval if and only if it is 3-convex at every point of the interval (see [1]).

Theorem 2.2. *Let $\mathcal{I}$ is an interval carrying a positive Borel measure ℓ and $\mathcal{A}$; $\mathcal{B}$; $\mathcal{C}$; $\mathcal{P}$; $\mathcal{Q}$; $\mathcal{R}$ are six disjoint compact subintervals of $\mathcal{I}$ of positive measure such that*

$$\ell(\mathcal{B}) = \ell(\mathcal{A}) + \ell(\mathcal{C}) \ \ ; \ \ \ell(\mathcal{Q}) = \ell(\mathcal{P}) + \ell(\mathcal{R}) \tag{2. 4}$$

and

$$\int_{\mathcal{B}} \alpha d\ell(\alpha) = \int_{\mathcal{A}} \alpha d\ell(\alpha) + \int_{\mathcal{C}} \alpha d\ell(\alpha) \ \ ; \ \ \int_{\mathcal{Q}} \beta d\ell(\beta) = \int_{\mathcal{P}} \beta d\ell(\beta) + \int_{\mathcal{R}} \beta d\ell(\beta). \tag{2. 5}$$

and also $c \in I^{\circ}$ is such that

$$\begin{aligned} &\max\{right\ end\ points\ of\ interval\ \mathcal{A}, \mathcal{B}, \mathcal{C}\} \\ &\qquad \le c \le \\ &\qquad \min\{left\ end\ points\ of\ interval\ \mathcal{P}, \mathcal{Q}, \mathcal{R}\} \end{aligned} \tag{2. 6}$$

Now, if

$$\int_{\mathcal{A}} \alpha^2 d\ell(t) + \int_{\mathcal{C}} \alpha^2 d\ell(\alpha) - \int_{\mathcal{B}} \alpha^2 d\ell(\alpha) = \int_{\mathcal{P}} \beta^2 d\ell(\beta) + \int_{\mathcal{R}} \beta^2 d\ell(\beta) - \int_{\mathcal{Q}} \beta^2 d\ell(\beta), \tag{2. 7}$$

then following inequality

$$\begin{aligned} &\int_{\mathcal{A}} f(\alpha) d\ell(\alpha) + \int_{\mathcal{C}} f(\alpha) d\ell(\alpha) - \int_{\mathcal{B}} f(\alpha) d\ell(\alpha) \\ &\qquad \le \int_{\mathcal{P}} f(\beta) d\ell(\beta) + \int_{\mathcal{R}} f(\beta) d\ell(\beta) - \int_{\mathcal{Q}} f(\beta) d\ell(\beta) \end{aligned} \tag{2. 8}$$

holds for every 3-convex function $f : \mathcal{I} \to \mathbb{R}$ at a point c.

Proof. Since f is 3-convex function at point $c \in \mathcal{I}^{\circ}$, then we have a constant K such that $F(\alpha) = f(\alpha) - \frac{K}{2}\alpha^2$ is concave on $\mathcal{I} \cap (-\infty, c]$. Therefore, for $\mathcal{A}$; $\mathcal{B}$; $\mathcal{C}$, three disjoint compact subintervals of $\mathcal{I} \cap [c, \infty)$ of positive measure, so by reverse of the inequality (1. 3), we have

$$\begin{aligned} 0 &\ge \int_{\mathcal{A}} F(\alpha) d\ell(\alpha) + \int_{\mathcal{C}} F(\alpha) d\ell(\alpha) - \int_{\mathcal{B}} F(\alpha) d\ell(\alpha) \\ &= \int_{\mathcal{A}} f(\alpha) d\ell(\alpha) + \int_{\mathcal{C}} f(\alpha) d\ell(\alpha) - \int_{\mathcal{B}} f(\alpha) d\ell(\alpha) \\ &- \frac{K}{2}\big(\int_{\mathcal{A}} \alpha^2 d\ell(\alpha) + \int_{\mathcal{C}} \alpha^2 d\ell(\alpha) - \int_{\mathcal{B}} \alpha^2 d\ell(\alpha)\big) \end{aligned}$$

Also, by using the fact that $F(\beta) = f(\beta) - \frac{K}{2}\beta^2$ is convex on $\mathcal{I} \cap [c, \infty)$. Therefore, for $\mathcal{P}$; $\mathcal{Q}$; $\mathcal{R}$, three disjoint compact subintervals of $\mathcal{I} \cap [c, \infty)$ of positive measure, so by use of the inequality (1. 3), we have

$$\begin{aligned} 0 &\le \int_{\mathcal{P}} F(\beta) d\ell(\beta) + \int_{\mathcal{R}} F(\beta) d\ell(\beta) - \int_{\mathcal{Q}} F(\beta) d\ell(\beta) \\ &= \int_{\mathcal{P}} f(\beta) d\ell(\beta) + \int_{\mathcal{R}} f(\beta) d\ell(\beta) - \int_{\mathcal{Q}} f(\beta) d\ell(\beta) \\ &- \frac{K}{2}\big(\int_{\mathcal{P}} \beta^2 d\ell(\beta) + \int_{\mathcal{R}} \beta^2 d\ell(\beta) - \int_{\mathcal{Q}} \beta^2 d\ell(\beta)\big) \end{aligned}$$

From above, we have

$$\int_{\mathcal{A}} f(\alpha)d\ell(\alpha) + \int_{\mathcal{C}} f(\alpha)d\ell(\alpha) - \int_{\mathcal{B}} f(\alpha)d\ell(\alpha) - \frac{K}{2}\big(\int_{\mathcal{A}} \alpha^2 d\ell(\alpha) + \int_{\mathcal{C}} \alpha^2 d\ell(\alpha) - \int_{\mathcal{B}} \alpha^2 d\ell(\alpha)\big) \leq 0 \leq$$

$$\int_{\mathcal{P}} f(\beta)d\ell(\beta) + \int_{\mathcal{R}} f(\beta)d\ell(\beta) - \int_{\mathcal{Q}} f(\beta)d\ell(\beta) - \frac{K}{2}\big(\int_{\mathcal{P}} \beta^2 d\ell(\beta) + \int_{\mathcal{R}} \beta^2 d\ell(\beta) - \int_{\mathcal{Q}} \beta^2 d\ell(\beta)\big).$$

So,

$$\begin{aligned}&\int_{\mathcal{A}} f(\alpha)d\ell(\alpha) + \int_{\mathcal{C}} f(\alpha)d\ell(\alpha) - \int_{\mathcal{B}} f(\alpha)d\ell(\alpha) \\ &\quad - \frac{K}{2}\big(\int_{\mathcal{A}} \alpha^2 d\ell(\alpha) + \int_{\mathcal{C}} \alpha^2 d\ell(\alpha) - \int_{\mathcal{B}} \alpha^2 d\ell(\alpha)\big) \\ &\quad \leq \int_{\mathcal{P}} f(\beta)d\ell(\beta) + \int_{\mathcal{R}} f(\beta)d\ell(\beta) - \int_{\mathcal{Q}} f(\beta)d\ell(\beta) \\ &\quad - \frac{K}{2}\big(\int_{\mathcal{P}} \beta^2 d\ell(\beta) + \int_{\mathcal{R}} \beta^2 d\ell(\beta) - \int_{\mathcal{Q}} \beta^2 d\ell(\beta)\big). \qquad (2.\ 9)\end{aligned}$$

By using (2. 7), we get (2. 8). ∎

Remark 2.3. *From the proof of the Theorem 2.2, we have*

$$\int_{\mathcal{A}} f(\alpha)d\ell(\alpha) + \int_{\mathcal{C}} f(\alpha)d\ell(\alpha) - \int_{\mathcal{B}} f(\alpha)d\ell(\alpha) \leq \frac{K}{2}\big(\int_{\mathcal{A}} \alpha^2 d\ell(\alpha) + \int_{\mathcal{C}} \alpha^2 d\ell(\alpha) - \int_{\mathcal{B}} \alpha^2 d\ell(\alpha)\big) \qquad (2.\ 10)$$

and

$$\int_{\mathcal{P}} f(\beta)d\ell(\beta) + \int_{\mathcal{R}} f(\beta)d\ell(\beta) - \int_{\mathcal{Q}} f(\beta)d\ell(\beta) \geq \frac{K}{2}\big(\int_{\mathcal{P}} \beta^2 d\ell(\beta) + \int_{\mathcal{R}} \beta^2 d\ell(\beta) - \int_{\mathcal{Q}} \beta^2 d\ell(\beta)\big) \qquad (2.\ 11)$$

So under assumption (2. 7), we can get a improvement of (2. 8) as follow

$$\begin{aligned}&\int_{\mathcal{A}} f(\alpha)d\ell(\alpha) + \int_{\mathcal{C}} f(\alpha)d\ell(\alpha) - \int_{\mathcal{B}} f(\alpha)d\ell(\alpha) \\ &\quad \leq \frac{K}{2}\big(\int_{\mathcal{A}} \alpha^2 d\ell(\alpha) + \int_{\mathcal{C}} \alpha^2 d\ell(\alpha) - \int_{\mathcal{B}} \alpha^2 d\ell(\alpha)\big) \\ &\quad \big\{ = \frac{K}{2}\big(\int_{\mathcal{P}} \beta^2 d\ell(\beta) + \int_{\mathcal{R}} \beta^2 d\ell(\beta) - \int_{\mathcal{Q}} \beta^2 d\ell(\beta)\big)\big\} \\ &\quad \leq \int_{\mathcal{P}} f(\beta)d\ell(\beta) + \int_{\mathcal{R}} f(\beta)d\ell(\beta) - \int_{\mathcal{Q}} f(\beta)d\ell(\beta) \qquad (2.\ 12)\end{aligned}$$

Now, we give next result which weakens the assumption (2. 7) such that inequality (2. 8) holds under this new condition.

Theorem 2.4. *Suppose that $\mathcal{I}$ is an interval carrying a positive Borel measure ℓ and $\mathcal{A}$; $\mathcal{B}$; $\mathcal{C}$; $\mathcal{P}$; $\mathcal{Q}$; $\mathcal{R}$ are six disjoint compact subintervals of $\mathcal{I}$ of positive measure such that (2. 4) and (2. 5) hold with*

$$a = \max\{right\ end\ points\ of\ interval\ \mathcal{A},\mathcal{B},\mathcal{C}\} \leq \min\{left\ end\ points\ of\ interval\ \mathcal{P},\mathcal{Q},\mathcal{R}\} = b \quad (2.\ 13)$$

and $f \to \mathbb{R}$ is 3-convex at a point c for some $c \in [a,b]$. Then if
(a)

$$f''_{-}(a)\ \geq\ 0$$

and

$$\int_{\mathcal{A}} \alpha^2 d\ell(\alpha) + \int_{\mathcal{C}} \alpha^2 d\ell(\alpha) - \int_{\mathcal{B}} \alpha^2 d\ell(\alpha) \leq \int_{\mathcal{P}} \beta^2 d\ell(\beta) + \int_{\mathcal{R}} \beta^2 d\ell(\beta) - \int_{\mathcal{Q}} \beta^2 d\ell(\beta)$$

or
(b)

$$f''_{+}(b)\ \leq\ 0$$

and

$$\int_{\mathcal{A}} \alpha^2 d\ell(\alpha) + \int_{\mathcal{C}} \alpha^2 d\ell(\alpha) - \int_{\mathcal{B}} \alpha^2 d\ell(\alpha) \geq \int_{\mathcal{P}} \beta^2 d\ell(\beta) + \int_{\mathcal{R}} \beta^2 d\ell(\beta) - \int_{\mathcal{Q}} \beta^2 d\ell(\beta)$$

or
(c)

$$f''_{-}(a)\ <\ 0\ <\ f''_{+}(b)\ \ and\ \ f\ is\ 3-convex,$$

then (2. 8) holds.

Proof. The idea of proof is similar to proof of Theorem 2.2. Hence, by proceeding as in the proof of Theorem 2.2. From the inequality 2. 9 , we have

$$\begin{aligned} \frac{K}{2}\big[\big(\int_{\mathcal{P}} \beta^2 d\ell(\beta) + \int_{\mathcal{R}} \beta^2 d\ell(\beta) - \int_{\mathcal{Q}} \beta^2 d\ell(\beta)\big) \\ - \big(\int_{\mathcal{A}} \alpha^2 d\ell(\alpha) + \int_{\mathcal{C}} \alpha^2 d\ell(\alpha) - \int_{\mathcal{B}} \alpha^2 d\ell(\alpha)\big)\big] \\ \leq \int_{\mathcal{P}} f(\beta) d\ell(\beta) + \int_{\mathcal{R}} f(\beta) d\ell(\beta) - \int_{\mathcal{Q}} f(\beta) d\ell(\beta) \\ - \big(\int_{\mathcal{A}} f(\alpha) d\ell(\alpha) + \int_{\mathcal{C}} f(\alpha) d\ell(\alpha) - \int_{\mathcal{B}} f(\alpha) d\ell(\alpha)\big) \end{aligned}$$

Now, due to the concavity of F on $\mathcal{I} \cap (-\infty, c]$ and convexity of F on $\mathcal{I} \cap [c, \infty)$, so for every distinct points $\alpha_j \in \mathcal{I} \cap (-\infty, a]$ and $\beta_j \in \mathcal{I} \cap [b, \infty)$, $j = 1, 2, 3$, we have

$$[\alpha_1, \alpha_2, \alpha_3]f \leq K \leq [\beta_1, \beta_2, \beta_3]f$$

Letting $\alpha_j \nearrow a$ and $\beta_j \searrow b$, we get (if exists)

$$f''_{-}(a) \leq K \leq f''_{+}(b)$$

Therefore, if assumptions (a) or (b) holds, then

$$\frac{K}{2}\big[\big(\int_{\mathcal{P}}\beta^2 d\ell(\beta)+\int_{\mathcal{R}}\beta^2 d\ell(\beta)-\int_{\mathcal{Q}}\beta^2 d\ell(\beta)\big)-\big(\int_{\mathcal{A}}\alpha^2 d\ell(\alpha)+\int_{\mathcal{C}}\alpha^2 d\ell(\alpha)-\int_{\mathcal{B}}\alpha^2 d\ell(\alpha)\big)\big]$$

is positive and we conclude the result. If the assumption (c) holds, the f''_- is left continuous, f''_+ is right continuous, they are both non-decreasing and $f''_- \leq f''_+$. Therefore, there exists $\tilde{c} \in [a, b]$ such that f with associated constant $\tilde{K} = 0$ and again, we can deduce the result. ∎

Remark 2.5. *Again from the proof of Theorem 2.4, we obtain the inequalities (2. 10) and (2. 11). Now, under assumption (a), (b) or (c) of Theorem 2.4, K is positive or negative or zero respectively due to argument discussed in the proof. Therefore, we get a better improvement of (2. 8) then (2. 12) in this case as follow*

$$\begin{aligned}\int_{\mathcal{A}} f(\alpha)d\ell(\alpha) &+ \int_{\mathcal{C}} f(\alpha)d\ell(\alpha) - \int_{\mathcal{B}} f(\alpha)d\ell(\alpha)\\ &\leq \frac{K}{2}\big(\int_{\mathcal{A}}\alpha^2 d\ell(\alpha)+\int_{\mathcal{C}}\alpha^2 d\ell(\alpha)-\int_{\mathcal{B}}\alpha^2 d\ell(\alpha)\big)\\ &\leq \frac{K}{2}\big(\int_{\mathcal{P}}\beta^2 d\ell(\beta)+\int_{\mathcal{R}}\beta^2 d\ell(\beta)-\int_{\mathcal{Q}}\beta^2 d\ell(\beta)\big)\\ &\leq \int_{\mathcal{P}} f(\beta)d\ell(\beta)+\int_{\mathcal{R}} f(\beta)d\ell(\beta)-\int_{\mathcal{Q}} f(\beta)d\ell(\beta) \quad (2.\ 14)\end{aligned}$$

Under the assumption of Theorem 2.2 with $f : \mathcal{I} \to \mathbb{R}$ is 3-concave at point $c \in \mathcal{I}^\circ$, the reverse of inequality (2. 8) holds. Now, we give only the statement of the theorem with weaker condition which can be proved in similar way under which the reverse of inequality (2. 8) holds for $f : \mathcal{I} \to \mathbb{R}$ is 3-concave at point $c \in \mathcal{I}^\circ$.

Theorem 2.6. *Suppose that $\mathcal{I}$ is an interval carrying a positive Borel measure ℓ and $\mathcal{A}$; $\mathcal{B}$; $\mathcal{C}$; $\mathcal{P}$; $\mathcal{Q}$; $\mathcal{R}$ are six disjoint compact subintervals of $\mathcal{I}$ of positive measure such that (2. 4) and (2. 5) hold with*

$$\begin{aligned}a = \max\{right\ end\ points\ of\ interval\ \mathcal{A},\mathcal{B},\mathcal{C}\}\\ \leq \min\{left\ end\ points\ of\ interval\ \mathcal{P},\mathcal{Q},\mathcal{R}\} = b \quad (2.\ 15)\end{aligned}$$

and $f : \mathcal{I} \to \mathbb{R}$ is 3-concave at a point c for some $c \in [a, b]$. Then if
(a)

$$f''_-(a) \ \leq\ 0$$

and

$$\begin{aligned}\int_{\mathcal{A}}\alpha^2 d\ell(\alpha) &+ \int_{\mathcal{C}}\alpha^2 d\ell(\alpha) - \int_{\mathcal{B}}\alpha^2 d\ell(\alpha)\\ &\geq \int_{\mathcal{P}}\beta^2 d\ell(\beta)+\int_{\mathcal{R}}\beta^2 d\ell(\beta)-\int_{\mathcal{Q}}\beta^2 d\ell(\beta)\end{aligned}$$

or
(b)

$$f''_+(b) \ \geq\ 0$$

and

$$\int_{\mathcal{A}} \alpha^2 d\ell(\alpha) + \int_{\mathcal{C}} \alpha^2 d\ell(\alpha) - \int_{\mathcal{B}} \alpha^2 d\ell(\alpha)$$
$$\leq \int_{\mathcal{P}} \beta^2 d\ell(\beta) + \int_{\mathcal{R}} \beta^2 d\ell(\beta) - \int_{\mathcal{Q}} \beta^2 d\ell(\beta)$$

or

(c)

$$f''_-(a) \ < \ 0 \ < \ f''_+(b) \ \ and \ \ f \ is \ 3-concave,$$

then reverse of (2. 8) holds.

Remark 2.7. *Similarly as in Remark 2.5, we obtain the reverse of inequalities (2. 10) and*

(2. 11) from the proof of Theorem 2.6. Now, due the convexity of F on $\mathcal{I} \cap (-\infty, c]$ and concavity of F on $\mathcal{I} \cap [c, \infty)$, so for every distinct points $\tilde{\alpha_j} \in \mathcal{I} \cap (-\infty, a]$ and $\tilde{\beta}_j \in \mathcal{I} \cap [b, \infty)$, $j = 1, 2, 3.$, we have

$$[\tilde{\alpha_1}, \tilde{\alpha_2}, \tilde{\alpha_3}]f \geq K \geq [\tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3]f$$

Letting $\tilde{\alpha}_j \nearrow a$ and $\tilde{\beta}_j \searrow b$, we get (if exists)

$$f''_-(a) \geq K \geq f''_+(b)$$

Now, under assumption (a) or (b) or (c) of Theorem 2.6, K is negative or positive or zero respectively due to argument discussed above. Therefore, we get a better improvement in this case as follow

$$\begin{aligned}
&\int_{\mathcal{A}} f(\alpha) d\ell(\alpha) + \int_{\mathcal{C}} f(\alpha) d\ell(\alpha) - \int_{\mathcal{B}} f(\alpha) d\ell(\alpha) \\
&\geq \frac{K}{2}\Big(\int_{\mathcal{A}} \alpha^2 d\ell(\alpha) + \int_{\mathcal{C}} \alpha^2 d\ell(\alpha) - \int_{\mathcal{B}} \alpha^2 d\ell(\alpha)\Big) \\
&\geq \frac{K}{2}\Big(\int_{\mathcal{P}} \beta^2 d\ell(\beta) + \int_{\mathcal{R}} \beta^2 d\ell(\beta) - \int_{\mathcal{Q}} \beta^2 d\ell(\beta)\Big) \\
&\geq \int_{\mathcal{P}} f(\beta) d\ell(\beta) + \int_{\mathcal{R}} f(\beta) d\ell(\beta) - \int_{\mathcal{Q}} f(\beta) d\ell(\beta) \qquad (2.\ 16)
\end{aligned}$$

Theorem 2.8. *Suppose that $\mathcal{I}$ is an interval carrying a Borel measure ℓ and $\mathcal{A}$; $\mathcal{B}$; $\mathcal{C}$; $\mathcal{P}$; $\mathcal{Q}$; $\mathcal{R}$ are six disjoint subintervals of $\mathcal{I}$ with the restriction of ℓ to each of the intervals $\mathcal{A}, \mathcal{C}, \mathcal{P}$ and $\mathcal{R}$ is a Steffensen-Popoviciu measure and the restriction of ℓ to $\mathcal{B}$ and $\mathcal{Q}$ is a dual Steffensen-Popoviciu measure such that*

$$\ell(\mathcal{B}) = \ell(\mathcal{A}) + \ell(\mathcal{C}) \ \ ; \ \ \ell(\mathcal{Q}) = \ell(\mathcal{P}) + \ell(\mathcal{R}) \qquad (2.\ 17)$$

and

$$\int_{\mathcal{B}} \alpha d\ell(\alpha) = \int_{\mathcal{A}} \alpha d\ell(\alpha) + \int_{\mathcal{C}} \alpha d\ell(\alpha) \ \ ; \ \ \int_{\mathcal{Q}} \beta d\ell(\beta) = \int_{\mathcal{P}} \beta d\ell(\beta) + \int_{\mathcal{R}} \beta d\ell(\beta). \qquad (2.\ 18)$$

and also $c \in \mathcal{I}^{\circ}$ is such that

$$\begin{aligned}
&\max\{right\ end\ points\ of\ interval\ \mathcal{A}, \mathcal{B}, \mathcal{C}\} \\
&\leq c \leq \\
&\min\{left\ end\ points\ of\ interval\ \mathcal{P}, \mathcal{Q}, \mathcal{R}\} \qquad (2.\ 19)
\end{aligned}$$

Now, if

$$\int_{\mathcal{A}} \alpha^2 d\ell(\alpha) + \int_{\mathcal{C}} \alpha^2 d\ell(\alpha) - \int_{\mathcal{B}} \alpha^2 d\ell(\alpha) = \int_{\mathcal{P}} \beta^2 d\ell(\beta) + \int_{\mathcal{R}} \beta^2 d\ell(\beta) - \int_{\mathcal{Q}} \beta^2 d\ell(\beta), \tag{2. 20}$$

then following inequality

$$\int_{\mathcal{A}} f(\alpha) d\ell(\alpha) + \int_{\mathcal{C}} f(\alpha) d\ell(\alpha) - \int_{\mathcal{B}} f(\alpha) d\ell(\alpha) \leq \int_{\mathcal{P}} f(\beta) d\ell(\beta) + \int_{\mathcal{R}} f(\beta) d\ell(\beta) - \int_{\mathcal{Q}} f(\beta) d\ell(\beta) \tag{2. 21}$$

holds for every $f : \mathcal{I} \to \mathbb{R}$ 3*-convex function at a point* c*.*

Remark 2.9. *The statement of Theorem 2.8 can be weakened under the similar setting as given in Theorem 2.4 such that the inequality (2. 21) holds.*

3. CONCLUSION

In this paper, we have studied the class of 3-convex/concave functions at a point which is larger then that of 3-convex/concave functions and have developed some interesting techniques for this new class. Using these techniques, we have established similar results to G. Bennett' inequality for this class. Methods developed and results proved in this paper may stimulate further research in this field.
The interested researchers are encouraged to find the particular examples of the results presented in this paper.

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