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Expanding the Convergence Domain of Newton–like Methods and Applications in Banach Space

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Abstract. We expand the convergence domain of Newton–like methods for solving nonlinear equations in a Banach space setting. Using more precise majorizing sequences, we provide a more precise local as well as a semilocal convergence analysis than in earlier studies such as [2]-[14], [16]-[30]. Our results are illustrated by several numerical examples, for which older convergence conditions do not hold but for which our convergence conditions are satisfied.

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1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution of the equation

$$\mathcal{J}\,(F(x) + G(x)) = 0, \tag{1. 1}$$

where F is a Fréchet–differentiable operator defined on an open convex subset of a Banach space $\mathcal{X}$ with values in Banach space , $\mathcal{J} \in (,\mathcal{X})$ the space of bounded linear operators from $\mathcal{X}$ into and $G\ :\longrightarrow$ is continuous.

Many problems from Applied sciences such as engineering, optimization, economics, physics, mathematical bilogy and other disciplines can be formulated like equation (1. 1) using mathematical modelling [6],[8], [11],[15], [21], [25], [26]. The solutions of these equations can be found in closed form only in special cases. That is why the solution methods for these equations are iterative– when starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework. We consider Newton–like methods

$$x_{n+1} = x_n - A(x_n)^{\#}\ (F(x_n) + G(x_n)) \text{ for each } n = 0, 1, 2, \cdots, \tag{1. 2}$$

to generate a sequence approximating , where $x_0 \in D$ is an initial point. Here, $F'(x_n)$ denotes the Fréchet–derivative operator F evaluated at $x = x_n$, $A(x_n) \in (\mathcal{X},)$ is an approximation of $F'(x_n)$ and $A(x_n)^{\#}$ denotes an outer inverse of $A(x_n)$,
i.e., $A(x_n)^{\#}A(x_n)A(x_n)^{\#} = A(x_n)^{\#}$ $(n \geq 0)$ [4, 6, 8, 11], [20, 12], [28].

Under some Lipschitz–type assumptions, Rheinboldt [27] presented a convergence theorem for (1. 2), when $A(x_n)^{\#} = A(x_n)^{-1}$ $(n \geq 0)$ and $G(x) = 0$ on , which includes the Newton–Kantorovich theorem for Newton's method (i.e., $A(x_n) = F'(x_n)$) as a special case. A further generalization was given by Dennis [14]. Yamamoto [18], Argyros and Hilout [10]) and others [1, 2, 16, 28, 3] improved on the error bounds obtained by the above. In the context of outer and generalized inverses, Ben–Israel [20], Deuflhard [15], Häubler [28], Yamamoto [29], Chen and Nashed [12] and Argyros [4, 5, 6], have established Newton–Kantorovich–type theorems under various conditions.

Our goal is to expand the applicability of Newton–like methods (1. 2) for solving equations by enlarging the convergence domain of these methods. Using more precise majorizing sequences we provide a tighter convergence analysis than in earlier studies such as [2]-[14], [16]-[30]. Moreover, our sufficient semilocal convergence conditions are also weaker. Numerical examples where older convergence conditions do not hold but for which our convergence conditions are satisfied in this study.

The study about convergence matter of iterative procedures is usually based on two types: semilocal and local convergence analysis. The semilocal convergence matter is, based on the informatiuon around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. The paper is organized as follows.

Section 2 contains the semilocal and local convergence of Newton–like method where as the examples are given in the concluding Section 3.

2. Semilocal convergence analysis

We need the following auxiliary result on the convergence of majorizing sequences for Newton–like method. Let $K_0 > 0$, $K > 0$, $L \geq 0$, $L_0 \geq 0$, $\ell_0 \geq 0$, $\ell \geq 0$, $M \geq 0$, $\lambda \geq 0$, $\mu \geq 0$ and $\eta > 0$ be given prameters. Define parameter α by

$$\alpha = \frac{2\,(K - 2\,M)}{K + \sqrt{K^2 + 8\,L\,(K - 2\,M)}}. \tag{2. 3}$$

Suppose the following conditions hold

$$2\,M < K,\ \mu \leq \alpha(1 - \ell); \tag{2. 4}$$

$$L_0\eta + \ell_0 < 1,\ \ Lt_2 + \ell < 1; \tag{2. 5}$$

and

$$(M + \alpha L)[\frac{t_2 - t_1}{1 - \alpha} + \eta] + \mu \leq \alpha(1 - \ell), \tag{2. 6}$$

where $t_1,\ t_2$ are given in (2. 7). α_0 is given in (2. 11) and $\alpha_0 \leq \alpha$. Then, scalar sequence $\{t_n\}$ given by

$$\begin{aligned} & t_0 = 0, \quad t_1 = \eta, \qquad t_2 = \eta + \tfrac{K_0\,\eta + 2\,\lambda}{2\,(1-\ell_0 - L_0\,\eta)}\,\eta \\ & t_{n+2} = t_{n+1} + \frac{K\,(t_{n+1}-t_n) + 2\,(M\,t_n + \mu)}{2\,(1-\ell - L\,t_{n+1})}\,(t_{n+1}-t_n) \end{aligned} \tag{2. 7}$$

for each $n = 1, 2, \cdots$, is well defined, increasing, bounded from above by

$$= \left(1 + \frac{K_0\,\eta + 2\,\lambda}{2\,(1-\ell_0 - L_0\,\eta)\,(1-\alpha)}\right)\eta \tag{2. 8}$$

and converges to its unique least upper bound which satisfies

$$\in [t_2,]. \tag{2. 9}$$

Moreover, the following estimates hold for each $n = 1, 2, \cdots$:

$$0 < t_{n+2} - t_{n+1} \leq \frac{K_0\,\eta + 2\,\mu}{2\,(1-\ell - L\,\eta)}\,\alpha^n\,\eta. \tag{2. 10}$$

Proof. Parameter α belongs in interval $(0, 1)$ by (2. 3) and (2. 4). Let

$$\alpha_0 = \frac{K\,(t_2 - t_1) + 2\,(M\,t_1 + \mu)}{2\,(1-\ell - L\,t_2)}. \tag{2. 11}$$

It follows from (2. 5) and (2. 7) that

$$L\,t_2 + \ell < 1. \tag{2. 12}$$

Moreover, by (2. 3), (2. 6) and (2. 12), we have that

$$0 < \alpha_0 \leq \alpha. \tag{2. 13}$$

We shall show using induction on the integer $i \geq 1$:

$$0 \leq \frac{K\,(t_{i+1} - t_i) + 2\,(M\,t_i + \mu)}{2\,(1-\ell - L\,t_{i+1})} \leq \alpha. \tag{2. 14}$$

Estimate (2. 14) is true for $i = 1$ by (2. 13). Then, we have by (2. 7) and (2. 14) for $i = 1$ that

$$\begin{aligned} 0 < t_3 - t_2 \leq \alpha\,(t_2 - t_1) \quad & \Longrightarrow \quad t_3 \leq t_2 + \alpha\,(t_2 - t_1) \\ & \Longrightarrow \quad t_3 \leq t_2 + (1+\alpha)\,(t_2 - t_1) - (t_2 - t_1) \\ & \Longrightarrow \quad t_3 \leq t_1 + \tfrac{1-\alpha^2}{1-\alpha}\,(t_2 - t_1) < . \end{aligned} \tag{2. 15}$$

Assume (2. 14) holds for all $j \leq i$. Then, we have that

$$0 < t_{j+2} - t_{j+1} \leq \alpha^j\,(t_2 - t_1) \tag{2. 16}$$

and

$$t_{j+2} \leq t_1 + \frac{1-\alpha^{j+1}}{1-\alpha}\,(t_2 - t_1). \tag{2. 17}$$

We must show

$$0 \leq \frac{K\,(t_{j+2} - t_{j+1}) + 2\,(M\,t_j + \mu)}{2\,(1-\ell - L\,t_{j+1})} \leq \alpha \tag{2. 18}$$

or

$$K\,(t_{j+2} - t_{j+1}) + 2\,(M\,t_j + \mu) \leq 2\,\alpha\,(1-\ell - L\,t_{j+1})$$

or

$$\begin{aligned} & K\,(t_2 - t_1)\,\alpha^j + 2\,M\,\tfrac{1-\alpha^j}{1-\alpha}\,(t_2 - t_1) + 2\,\alpha\,L\,\tfrac{1-\alpha^{j+1}}{1-\alpha}\,(t_2 - t_1) + \\ & 2\,(M\,\eta + \alpha\,L\,\eta + \mu - \alpha\,(1-\ell)) \leq 0. \end{aligned} \tag{2. 19}$$

Estimate (2. 19) motivates us to define recurrent functions on $(0,1)$ by

$$\begin{aligned} f_j(s) &= K\,(t_2-t_1)\,s^j + 2\,M\,(t_2-t_1)\,(1+s+s^2+\cdots+s^{j-1}) \\ & 2\,L\,(t_2-t_1)\,s\,(1+s+\cdots+s^j) + 2\,((M+s\,L)\,\eta + \mu - s\,(1-\ell)). \end{aligned} \tag{2. 20}$$

We need a relationship between two consecutive functions f_j:

$$f_{j+1}(s) = f_{j+1}(s) - f_j(s) + f_j(s) = f_j(s) + g(s)\,(t_2-t_1)\,s^j, \tag{2. 21}$$

where

$$g(s) = 2\,L\,s^2 + K\,s + 2\,M - K. \tag{2. 22}$$

Note that by (2. 4) and (2. 22), function g has an unique positive zero α given by (2. 3). In view of (2. 20), estimate (2. 19) holds if

$$f_j(\alpha) \le 0. \tag{2. 23}$$

Define functions f_∞ on $(0,1)$ by

$$f_\infty(s) = \lim_{j\to\infty} f_j(s). \tag{2. 24}$$

We have from (2. 21) and (2. 22) that

$$f_{j+1}(\alpha) = f_j(\alpha). \tag{2. 25}$$

Then, it follows from (2. 24) and (2. 25) that (2. 23) holds if

$$f_\infty(\alpha) \le 0. \tag{2. 26}$$

By letting $j \longrightarrow \infty$ in (2. 16) and by (2. 6), we have that

$$f_\infty(\alpha) = 2\left(M\,\eta + \alpha\,L\,\eta + \mu - \alpha\,(1-\ell) + \frac{M}{1-\alpha}\,(t_2-t_1) + \frac{\alpha\,L}{1-\alpha}\,(t_2-t_1)\right) \le 0. \tag{2. 27}$$

The induction for (2. 14) is now completed. Hence, sequence $\{t_n\}$ is increasing, bounded from above by given in (2. 8) and as such it converges to its unique least upper bound which satisfies (2. 9). The proof of Lemma 2 is complete. ⊠

Inequalities (2. 5)–(2. 6) describe the smallness of η and can be solved for η (see, e.g. Section 3). However, we decided to leave them as uncluttered as possible, since their representation is very long. Notice also that these inequalities are the expected Kantorovich-type hypotheses appearing in these type of methods. Let $U(x,r)$ and $\overline{U}(x,r)$ stand, respectively, for the open and closed ball in $\mathcal{X}$ with center x and radius $r>0$.

We shall use the following conditions for the semilocal convergence of Newton–like method

(C) $F\ :\subseteq \mathcal{X} \to$ is Fréchet–differentiable and $G\ : \to$ is continuous;

There exist an approximation $A(x) \in L(X,Y)$ of $F'(x)$, an open convex subset $_0$ of , an initial point $x_0 \in_0$, a bounded outer inverse $A^{\#}$ of $A(x_0)$ and a parameter $\eta > 0$ such that

$$\parallel A^{\#}\,(F(x_0)+G(x_0)) \parallel \quad\le\quad \eta. \tag{2. 28}$$

Moreover, suppose that there exist parameters $K>0$, $M\ge 0$, $\mu_0\ge 0$, $\mu_2\ge 0$, $L\ge 0, \ell \ge 0$ such that for each x and y in $_0$

$$\begin{aligned} \parallel A^{\#}\,(F'(x+\theta(y-x)) - F'(x)) \parallel &\le K\theta \parallel x-y \parallel \text{ for each } \theta\in[0,1], && (2.\ 29)\\ \parallel A^{\#}\,(F'(x)-A(x)) \parallel &\le M \parallel x-x_0 \parallel + \mu_0, && (2.\ 30)\\ \parallel A^{\#}\,(G(x)-G(y)) \parallel &\le \mu_2 \parallel x-y \parallel, && (2.\ 31)\\ \parallel A^{\#}\,(A(x)-A(x_0)) \parallel &\le L \parallel x-x_0 \parallel + \ell; && (2.\ 32) \end{aligned}$$

hypotheses of Lemma 2 hold and

$$\overline{U}(x_0,) \subseteq_0, \tag{2. 33}$$

where is defined in Lemma 2.

In view of (2. 29), (2. 31) and (2. 32), we have respectively that for $x_1 = x_0 - A(x_0)^{\#}(F(x_0) + G(x_0))$ for each $x \in_0$

$$\|A^{\#}\,(F'(x_0+\theta(x_1-x_0)) - F'(x_0))\, \| \leq K_0\theta \,\|\, x_1 - x_0 \,\| \quad \text{for each } \theta \in [0,1], \tag{2. 34}$$

$$\|A^{\#}\,(G(x_1) - G(x_0))\, \| \leq \mu_1 \,\|\, x_1 - x_0 \,\|, \tag{2. 35}$$

and

$$\|A^{\#}\,(A(x_1) - A(x_0))\, \| \leq L_0 \,\|\, x_1 - x_0 \,\| + \ell_0. \tag{2. 36}$$

Clearly

$$K_0 \leq K,\ \mu_1 \leq \mu_2,\ L_0 \leq L,\ \ell_0 \leq \ell \tag{2. 37}$$

and $\frac{K}{K_0}$, $\frac{\mu_2}{\mu_1}$, $\frac{L}{L_0}$ can be arbitrarily large [5, 6, 8, 11]. Note that in practice the computation of K, μ_2, L and ℓ requires the computation of K, μ_1, L_0 and ℓ_0, respectively. So, (2. 34)-(2. 36) are not additional respectively to (2. 29), (2. 31) and (2. 32) hypotheses.

We can show the following semilocal convergence theorem for Newton–like method.

Theorem 1. *Suppose that the (C) conditions hold. Then, sequence $\{x_n\}$ $(n \geq 0)$ generated by Newton-like method is well defined, remains in $U(x_0,)$ for each $n = 0, 1, 2, \cdots$ and converges to a solution of equation $A^{\#}\,(F(x) + G(x)) = 0$ in $\overline{U}(x_0,)$. Moreover, the following estimates hold for all $n \geq 0$*

$$\|\, x_{n+1} - x_n \,\| \leq t_{n+1} - t_n \tag{2. 38}$$

and

$$\|\, x_n - \,\| \leq -t_n, \tag{2. 39}$$

where sequence $\{t_n\}$ $(n \geq 0)$ is given by (2. 7), with $\mu = \mu_0 + \mu_2$ and $\lambda = \mu_0 + \mu_1$. Furthemore, the solution of equation (1. 1) is unique in $\overline{U}(x_0,)$ provided that

$$\left(\frac{K}{2} + M + L\right) + \mu + \ell < 1. \tag{2. 40}$$

Proof. We shall show using induction on m that (2. 38) holds. Estimate (2. 39) will then follow from (2. 38) using standard majorization techniques [6, 8, 11, 21]. By the initial conditions, we have that

$$\|\, x_1 - x_0 \,\| \leq t_1 - t_0$$

and (2. 38) holds for $m = 0$. Using (2. 32), we get that

$$\|\, A^{\#}\,(A(x_m) - A)\, \| \leq L^* \,\|\, x_m - x_0 \,\| + \ell^* \leq L^*\, t_m + \ell^* \leq L^* \; + \ell^* < 1, \tag{2. 41}$$

where

$$L^* = \begin{cases} L_0, if\ m = 1 \\ L, if\ m = 2, 3, \cdots \end{cases}$$

and

$$\ell^* = \begin{cases} \ell_0, if\ m = 1 \\ \ell, if\ m = 2, 3, \cdots \end{cases}.$$

From Banach's perturbation Lemma [12, Lemma 2.2] and (2. 41), we obtain that $A(x_m)^{\#} := (I + A^{\#}\,(A(x_m) - A))^{-1}\,A^{\#}$ is an outer inverse of $A(x_m)$. Moreover, we have that

$$\|\, A(x_m)^{\#}\, A \,\| \leq (1 - L^* \,\|\, x_m - x_0 \,\| - \ell^*)^{-1} \leq (1 - L^*\, t_m - \ell^*)^{-1}$$

and $\mathcal{N}\,(A(x_m)^{\#}) = \mathcal{N}\,(A^{\#})$. Assume that for $1 \leq m \leq k$:

$$\| x_m - x_{m-1} \| \leq t_m - t_{m-1} \quad \text{and} \quad \mathcal{N}\,(A(x_{m-1})^{\#}) = \mathcal{N}\,(A^{\#}).$$

Then, we obtain that

$$\| x_m - x_0 \| \leq t_m - t_{m-1} \quad \text{and} \quad \mathcal{N}\,(A(x_m)^{\#}) = \mathcal{N}\,(A(x_{m-1})^{\#}) = \mathcal{N}\,(A^{\#}).$$

Hence, we have from [12, Lemma 2.3] that

$$A(x_m)^{\#}\,(I - A(x_{m-1})\,A(x_{m-1})^{\#}) = 0$$

and

$$\begin{aligned} x_{m+1} \quad - \quad & x_m = -A(x_m)^{\#}\,(F(x_m) + G(x_m)) \\ = \quad & -A(x_m)^{\#} \left(\int_0^1 (F'(x_m + t\,(x_{m-1} - x_m)) - F'(x_{m-1}))\,(x_m - x_{m-1})\,dt \right. \\ & \left. + (F'(x_{m-1}) - A(x_{m-1}))\,(x_m - x_{m-1}) + (G(x_m) - G(x_{m-1})) \right). \end{aligned} \tag{2. 42}$$

Using also [12, Lemma 2.3], we obtain that

$$A(x_m)^{\#}\,(I - A\;A^{\#}) = 0.$$

In view of (2. 29)–(2. 33), (2. 41) and (2. 42) for

$$K^{\star} = \begin{cases} K_0 & \text{if} \quad m = 1 \\ K & \text{if} \quad m = 2, 3, \cdots \end{cases}$$

we deduce that

$$\begin{aligned} & \| x_{m+1} - x_m \| \\ & \leq \| A(x_m)^{\#}\,A \| \; \left\{ \left(\int_0^1 \| A^{\#}\,(F'(x_m + t\,(x_{m-1} - x_m)) - F'(x_{m-1})) \| \; dt + \right. \right. \\ & \left. \left. \| A^{\#}\,(F'(x_{m-1}) - A(x_{m-1})) \| \right) \; \| x_m - x_{m-1} \| + \| A^{\#}\,(G(x_m) - G(x_{m-1})) \| \right\} \\ & \leq \frac{1}{1 - L^*\,t_k - \ell^*} \left(\frac{K^{\star}}{2} \; \| x_m - x_{m-1} \|^2 + (M \; \| x_{m-1} - x_0 \| + \mu) \; \| x_m - x_{m-1} \| \right) \\ & \leq \frac{1}{1 - L^*\,t_k - \ell^*} \left(\frac{K^{\star}}{2}\,(t_m - t_{m-1})^2 + M\,t_{m-1} + \mu \right) (t_m - t_{m-1}) = t_{m+1} - t_m, \end{aligned} \tag{2. 43}$$

which completes the induction. Hence, we have for any m that

$$\| x_{m+1} - x_m \| \leq t_{m+1} - t_m,$$

$$\| A^{\#}\,(A(x_{m+1}) - A) \| \leq L^* \; \| x_{m+1} - x_0 \| + \ell^* \leq L^*\,t_{m+1} + \ell^* \leq L^* \; + \ell^* < 1,$$

$$\| x_m - x_0 \| \leq t_m \leq$$

and $A(x_{m+1})^{\#} \;:=\; (I + A^{\#}\;(A(x_{m+1}) - A))^{-1}\,A^{\#}$ is an outer inverse of $A(x)$. It follows that $x_m \in U(x_0,)$, $m \geq 0$ and $\{x_m\}$ converges to a point in $\overline{U}(x_0,)$. The point is a solution of $A^{\#}\;(F(x) + G(x)) = 0$. Indeed, by definition

$$A(x_m)^{\#} = (I + A^{\#}\;(A(x_m) - A))^{-1}\;A^{\#}, \quad \text{for all} \quad m$$

and

$$\begin{aligned} 0 \quad = \quad & \lim_{m \longrightarrow \infty} (I + A^{\#}\,(A(x_m) - A))\,(x_m - x_{m-1}) \\ = \quad & \lim_{m \longrightarrow \infty} A^{\#}\,(F(x_m) + G(x_m)) = A^{\#}\,(F() + G()). \end{aligned}$$

Hence, solves equation $A^{\#}\ (F() + G()) = 0$. Finally to show that is the unique solution of equation (1. 1) in $\overline{U}(x_0,)$, as in (2. 42) and (2. 43), we get in turn for $\in \overline{U}(x_0,)$, with $A^{\#}\, (F() + G()) = 0$, the estimation

$$\begin{aligned}
&\parallel -x_{m+1} \parallel \\
&\leq \parallel A(x_m)^{\#}\, A \parallel \left\{ \left(\int_0^1 \parallel A^{\#}\, (F'(x_m + \theta\, (-x_m)) - F'(x_m)) \parallel\, d\theta \right.\right. \\
&\left.\left. + \parallel A^{\#}\, [F'(x_m) - A(x_m)] \parallel \right) \parallel -x_m \parallel + \parallel A^{\#}\, [G(x_m) - G()] \parallel \right\} \\
&\leq (1 - \ell^* - L^*\, t_{m+1})^{-1} \left(\frac{K^\star}{2} \parallel -x_m \parallel^2 + (M \parallel x_m - x_0 \parallel + \mu) \parallel -x_m \parallel \right) \\
&\leq (1 - \ell^* - L^*\, t_{m+1})^{-1} \left(\frac{K^\star}{2}\, (-t_m) + M\, t_m + \mu \right) \parallel -x_m \parallel \\
&\leq (1 - \ell^* - L^*\,)^{-1} \left(\frac{K^\star}{2}\, (-t_0) + M\, + \mu \right) \parallel -x_m \parallel < \parallel -x_m \parallel,
\end{aligned} \tag{2. 44}$$

by the uniqueness hypothesis (2. 40). It follows by (2. 44) that $\lim\limits_{m \longrightarrow \infty} x_m =$. But we showed $\lim\limits_{m \longrightarrow \infty} x_m =$. Hence, we deduce $=$. The proof of Theorem 1 is complete. ⊠

(a) The hypotheses of Lemma 2 are used to show that sequence $\{t_n\}$ is increasingly convergent. These conditions can be replaced in Theorem 1 by the weaker

$$t_n < \frac{1 - \ell}{L} \quad \text{for each} \quad n = 0, 1, 2, \cdots . \tag{2. 45}$$

It follows from (2. 7) and (2. 45) that sequence $\{t_n\}$ is increasing and bounded above by $\frac{1-\ell}{L}$ and as such it converges to some $t^* \in [t_2, \frac{1-\ell}{L}]$. Clearly, hypotheses of Lemma 2 imply (2. 45) but not necessarily vice versa. Then, (2. 45) can replace conditions of Lemma 2 in Theorem 1.

(b) Another set of conditions weaker than those of Lemma 2 but stronger than (2. 45) is given by the following

(H) Suppose that there exists $N \geq 1$ such that

$$t_0 < t_1 < \cdots < t_N < \frac{1 - \ell}{L} \text{ for } 0 \leq \ell < 1$$

and hypotheses of Lemma 2 are satisfied with $\eta_N = t_{N+1} - t_N$ replacing η. Then, sequence $\{t_n\}$ is increasingly convergent and

$$t_{n+2} - t_{n+1} \leq \alpha(t_{n+1} - t_n) \quad \text{for each } n = N, N+1, \cdots .$$

If $N = 1$ we obtain Lemma 2. Clearly, weaker hypotheses (H) can replace those of Lemma 2 in the hypotheses of Theorem 1.

The point can be replaced by , given in closed form by (2. 9) in all hypotheses of Theorem 1. As it was also noted in [17] (see Theorem 1 above), suppose that

$$[I + (A(x) - A(x_0))\, A^+]^{-1}\, A(x) \quad \text{maps } \mathcal{N}\,(A(x_0)) \text{ into } R\,(A(x_0)) \tag{2. 46}$$

whenever $I + (A(x) - A(x_0))\, A^+$ is invertible for some $x \in$, where $\mathcal{N}(A)$ denotes the null space of A and A^+ the generalized inverse of $A(x_0)$. Then by [12, Lemma 2.4]: $A(x_n) = [I + A^+(x_0)\, (A(x_n) - A(x_0))]^{-1}\, A^+(x_0)$ is a generalized inverse. Hence by [12, Lemma 2.4] and Theorem 1, we establish a semilocal convergence theorem for (1. 2) using generalized inverses. In the finite dimensional case ($\mathcal{X}$, both finite), condition (2. 46) can be replaced by

$$\text{rank } (A(x)) \leq \text{rank } (A(x_0)) \quad (x \in). \tag{2. 47}$$

Next, we present the local convergence of the Newton–like method. We shall use the same notation as in the semilocal case for the parameters involved. As in the semilocal convergence case we use the conditions

(C^*) $F\ :\subseteq \mathcal{X} \rightarrow$ is Fréchet–differentiable and $G\ :\rightarrow$ is continuous. There exist an approximation $A(x) \in L(X,Y)$ of $F'(x)$, an open convex subset $_0$ of , $x^* \in_0$, a bounded outer inverse $A_*^{\#}$ of $A(x^*)$ and parameters $K > 0$, $M \geq 0$, $\mu_0 \geq 0$, $\mu_2 \geq 0$, $L \geq 0, \ell \geq 0$ such that $A_*^{\#}(F(x^*)+G(x^*)) = 0$ and for each x in $_0$

$$\begin{aligned} \| A_*^{\#}\,(F'(x+\theta(x^*-x)) - F'(x)) \| &\leq K\theta\ \| x-x^* \| \ \text{ for each } \theta \in [0,1], \\ \| A_*^{\#}\,(F'(x) - A(x)) \| &\leq M\ \| x-x^* \| + \mu_0, \\ \| A_*^{\#}\,(G(x) - G(x^*)) \| &\leq \mu_2\ \| x-x^* \|, \\ \| A_*^{\#}\,(A(x) - A(x^*)) \| &\leq L\ \| x-x^* \| + \ell; \\ \mu+\ell &< 1, \end{aligned}$$

and

$$\overline{U}(x^*,R) \subseteq,$$

where $R = \frac{1-(\mu_0+\mu_2+\ell)}{\frac{K}{2}+M+L}$ and $\mu = \mu_0 + \mu_2$.

As in the case of the (C) conditions we have

$$\|A_*^{\#}\,(F'(x_1+\theta(x^*-x_1)) - F'(x_1)) \| \leq K_0\theta\ \| x_1-x^* \| \ \text{ for each } \theta \in [0,1],$$
$$\|A_*^{\#}\,(G(x_1) - G(x^*)) \| \leq \mu_1\ \| x_1-x^* \|,$$

and

$$\|A_*^{\#}\,(A(x_1) - A(x^*)) \| \leq L_0\ \| x_1-x^* \| + \ell_0.$$

Comments similar to the ones given for the semilocal convergence case can now follow.

Next, we present the local convergence result for the Newton–like method.

Theorem 2. *Suppose that the (C^*) conditions hold. Then, sequence $\{x_n\}$ $(n \geq 0)$ generated by Newton-like method for $x_0 \in U(x^*,R)$ is well defined, remains in $U(x^*,R)$ for each $n = 0,1,2,\cdots$ and converges to a solution x^*. Moreover, the following estimates hold*

$$\| x_{n+1}-x^* \| < a_n^*\|x_n-x^*\| \leq a\|x_n-x^*\|, \tag{2. 48}$$

and

$$a = \frac{(\frac{K^*}{2}+M)R+\mu}{1-(\ell+LR)},$$

where

$$a_n^* = \frac{\frac{K^*}{2}\|x_n-x^*+M\|x_n-x^*\|+\mu^*}{1-(\ell^*+L^*\|x_n-x^*\|)}$$

where K^*, L^* are given in Theorem 1,

$$\ell^* = \begin{cases} \ell_0, if\ n=0 \\ \ell, if\ n=1,2,3,\cdots \end{cases}$$

and

$$\mu^* = \begin{cases} \lambda, if\ n=0 \\ \mu, if\ n=1,2,3,\cdots \end{cases}.$$

Proof. Simply replace $x_0, A^{\#}, y^*$, (C), in (2. 44) respectively by $x^*, A_*^{\#}, x^*, (C^*)$ to obtain (2. 48) from which it follows $x_n \in \overline{U}(x^*,R)$ and $\lim_{n\to\infty} x_n = x^*$. That completes the proof of the Theorem.

3. SPECIAL CASES AND APPLICATIONS

In this Section we shall compare majorizing sequences with earlier ones.

Case 1 Let us introduce scalar sequence $\{s_n\}$ given by

$$s_0 = 0, \quad s_1 = \eta,$$
$$s_{n+2} = s_{n+1} + \frac{\overline{K}\,(s_{n+1} - s_n) + 2\,(M\ s_n + \mu)}{2\,(1 - \ell - L\,s_{n+1})}\,(s_{n+1} - s_n) \text{ for each } n = 0, 1, 2 \cdots \tag{3. 49}$$

where

$$\overline{K} = \begin{cases} K_0 & \text{if} \quad K_0 < K \\ K & \text{if} \quad K_0 = K. \end{cases}$$

Note that if $K_0 = K$, (3. 49) reduces to the majorizing sequence used in [2]-[11]. The majorizing sequence in [12, 13, 14, 28, 26, 27, 29, 30] is given by (for G=0)

$$v_0 = 0, \quad v_1 = \eta, \quad v_{n+2} = v_{n+1} + \frac{f(v_{n+1})}{q(v_{n+1})} \text{ for each } n = 0, 1, 2 \cdots \tag{3. 50}$$

where

$$f(s) = \frac{\sigma}{2}\,s^2 - (1 - b)\,s + \eta, \quad q(s) = 1 - L\,s - \ell,$$
$$b = \mu + \ell \quad \text{and} \quad \sigma = \max\{K, M + L\}.$$

Next, we compare majorizing sequence $\{s_n\}$ with $\{v_n\}$. (see [5, Proposision 3.1]) Suppose sequences $\{t_n\}$, $\{s_n\}$ and $\{v_n\}$ given by (2. 7), (3. 49) and (3. 50) respectively are increasingly convergent. Then, the following assertions hold

$$t_{n+1} \le s_{n+1} \quad \le \quad v_{n+1} \text{ for each } n = 1, 2, \cdots, \tag{3. 51}$$
$$t_{n+1} - t_n \le s_{n+1} - s_n \quad \le \quad v_{n+1} - v_n \text{ for each } n = 1, 2, \cdots, \tag{3. 52}$$
$$t^* - t_n \le -s_n \quad \le \quad v^\star - v_n \text{ for each } n = 0, 1, 2, \cdots, \tag{3. 53}$$
$$t^* \le\le v^\star, \tag{3. 54}$$

where $t^* = \lim_{n\to\infty} t_n$ and $= \lim_{n\to\infty} s_n$ and $= \lim_{n\to\infty} v_n$. Moreover, strict inequality holds in the right hand side inequality in (3. 51) and (3. 52) if $K < M + L$. Furthermore, strict inequality holds in the left hand side inequality in (3. 51) and (3. 52) if $\ell_0 < \ell$ or $L < K$.

So far we showed that $\{s_n\}$ is a tighter sequence than $\{v_n\}$ and the information on the location of the solution at least as precise, since $\le$. The sufficient convergence conditions given in this study can also be weaker in many interesting cases.

Case 2 We set $A^{\#}(x) = A(x)^{-1}$, $A(x) = F'(x)$ and $G(x) = 0$ on $_0$ for simplicity. The parameters are chosen to be $M = \mu = \ell = \ell_0 = \mu_0 = \lambda = \mu_1 = \mu_2 = 0$ and $K_0 = L_0 = L$. That is we consider the popular case of Newton's method. Then Lemma 2 reduces to:

Let $K > 0$, $L > 0$ and $\eta > 0$ be constants. Suppose $K \ge L$ and

$$h_2 = K_2\,\eta \le \frac{1}{2}, \tag{3. 55}$$

where $K_2 = \frac{1}{8}\,(4\,L + (K\,L + 8\,L^2)^{1/2} + (L\,K)^{1/2})$. Set

$$\alpha = \frac{2\,K}{K + (K^2 + 8\,L\,K)^{1/2}}. \tag{3. 56}$$

Then, scalar sequence $\{t_n\}$ given by

$$t_0 = 0, \quad t_1 = \eta, \quad t_2 = \eta + \tfrac{L\,\eta^2}{2\,(1-L\,\eta)}, \\ t_{n+2} = t_{n+1} + \frac{K\,(t_{n+1}-t_n)^2}{2\,(1-L\,t_{n+1})} \quad (n \geq 1), \tag{3. 57}$$

is well defined, increasing, bounded from above by

$$= \eta + \frac{L\,\eta^2}{2\,(1-\alpha)\,(1-L\,\eta)} \tag{3. 58}$$

and converges to its unique least upper bound which satisfies $\eta \leq \leq$. Moreover, the following estimates hold for all $n \geq 1$

$$0 < t_{n+2} - t_{n+1} \leq \alpha^n \, \frac{L\,\eta^2}{2\,(1-L\,\eta)}. \tag{3. 59}$$

Clearly (3. 55) is weakened even further if $K_0 < L$ or $L_0 < L$. The sufficient convergence condition in [4], [20]- [30] for $K_0 = L_0 = K = L$ is given by the Kantorovich condition

$$h_\star = K\,\eta \leq \frac{1}{2}. \tag{3. 60}$$

Moreover, if $L_0 = K_0 = K$, the condition given by us in [5]-[11] is

$$h_1 = K_1\,\eta \leq \frac{1}{2}, \tag{3. 61}$$

where $K_1 = \frac{1}{4}\,(K + 4\,L + (K^2 + 8\,L\,K)^{1/2})$. We have that

$$h_\star \leq \frac{1}{2} \Longrightarrow h_1 \leq \frac{1}{2} \Longrightarrow h_2 \leq \frac{1}{2}$$

but not necessarily vice versa unless if $K_0 = K$. Note also that since

$$L \leq K$$

holds in general and $\frac{L}{K}$ can be arbitrarily small, we have that

$$\frac{h_1}{h_\star} \longrightarrow \frac{1}{4}, \quad \frac{h_2}{h_\star} \longrightarrow 0, \quad \frac{h_2}{h_1} \longrightarrow 0 \quad \text{as} \quad \frac{L}{K} \longrightarrow 0.$$

Case 3 Let $A_n = \frac{1}{b_n} F'(x_n)$ for $b_n \neq 0$. That is we consider the damped Newton method [10]

$$x_{n+1} = x_n - b_n F'(x_n)^{\#}(F(x_n) + G(x_n)) \quad \text{for each } n - 0, 1, 2 \cdots .$$

Then, we can show to choose the parameters b_n. For example condition (2. 30) is satisfied provided that

$$|1 - \frac{1}{b_n}| L_0 \leq M$$

and

$$|1 - \frac{1}{b_n}| \|A^{\#} F'(x_0)\| \leq \mu_0,$$

since

$$\begin{aligned} \|A^{\#}(F'(x_n) - A(x_n)\| &= |1 - \frac{1}{b_n}| \|A^{\#} F'(x_n)\| \\ &\leq |1 - \frac{1}{b_n}| (\|A^{\#}(F'(x_n) - F'(x_0)) + A^{\#} F'(x_0)\| \end{aligned}$$

$$\leq \quad |1 - \frac{1}{b_n}|(L_0\|x_n - x_0\| + \|A^{\#}F'(x_0)\|.$$

Similarly, we find the other conditions on b_n using (2. 32) and (2. 36).

Case 4 Let $A_n = b_n I + B(x_n)$, where $B(x)$ is an approximation to $F'(x)$. That is we consider the Levenberg-Marquardt method (LM) [15, 16]

$$x_{n+1} = x_n - (b_n I + B(x_n))^{\#}(F(x_n) + G(x_n)) \quad \text{for each } n = 0, 1, 2, \cdots.$$

Then, the choice of b_n can be determined as in Case 3. As an example, if $B(x) = F'(x)$, then

$$\|A^{\#}(F'(x_n) - A(x_n)\| = \|A^{\#}b_n I\| = |b_n|,$$

so we can choose $M = 0$ and $|b_n| \leq \mu_0$. Notice that the conditions on parameters b_n are more general and weaker than the ones usually associated with the Damped Newton method or the Levenberg-Marquardt method.

Semi-local case.

(a) Let $G = 0$, $A(x)^{\#} = F'(x)^{-1}$, $x \in, \mathcal{X} == \mathbb{R}$, $x_0 = 1$ and $= \overline{U}(1, 1-a)$ for $a \in (0, .5)$. Define function F on by

$$F(x) = x^3 - a. \tag{3. 62}$$

Then, using (2. 28)-(2. 32) we get that $\eta = \frac{1-a}{3}$ and $K = 2\,(2-a)$. The Newton-Kantorovich condition (3. 60) is violated, since $h_\star = 4\,(1-a)\,(2-a)/3 > 1$ for each $a \in (0, .5)$. Hence, there is no guarantee under the Kantorovich theorem that sequence $\{x_n\}$ converges to . Using (2. 32), we get that $L = 3 - a$. Our hypothesis holds for $a = [.4271907643, .5)$. Hence, (3. 55) is violated, say for $a = .427$. However, hypotheses of Remark 2.3 (a) or (b) are satisfied.

(b) Let $G(x) = 3\varepsilon|x-1|$, $\varepsilon > 0$. Then, using (2. 31) we get $\mu_2 = \varepsilon$. So, the sufficient convergence condition given in [12, 13, 14, 30] is

$$2K\eta \leq (1-\varepsilon)^2$$

which is violated for all $\varepsilon > 0$. However, our conditions (see e.g. Lemma 2) are satisfied for sufficiently small ε.

Let $\mathcal{C}[0,1]$ stand for the space of continuous functions defined on interval $[0,1]$ and be equipped with the max-norm. Let also $\mathcal{X} == \mathcal{C}[0,1], = U(0,r)$ for some $r > 1$, $G = 0$ and $A(x)^{\#} = F'(x)^{-1}$. Define F on by

$$F(x)(s) = x(s) - y(s) - \zeta \int_0^1 G(s,t)\, x^3(t)\, dt, \quad x \in \mathcal{C}[0,1],\ s \in [0,1].$$

$y \in \mathcal{C}[0,1]$ is given, ζ is a real parameter and the Kernel G is the Green's function defined by

$$G(s,t) = \begin{cases} (1-s)\,t & if \quad t \leq s \\ s\,(1-t) & if \quad s \leq t. \end{cases}$$

Then, the Fréchet-derivative of F is defined by

$$(F'(x)\,(w))(s) = w(s) - 3\,\zeta \int_0^1 G(s,t)\, x^2(t)\, y(t)\, dt, \quad w \in \mathcal{C}[0,1],\ s \in [0,1].$$

Let us choose $x_0(s) = y(s) = 1$ and $|\zeta| < 8/3$. Then, we have that

$$\| -F'(x_0) \| < \frac{3}{8}\,|\zeta|, \quad F'(x_0)^{-1} \in (, \mathcal{X}),$$

r	$\zeta^\star$	$h_\star$	h_1	h_2
2.09899	.9976613778	1.007515200	.9757536188	.9639223786
2.19897	.9831766058	1.055505600	.9915015816	.9678118280
2.29597	.9698185659	1.102065600	1.006635036	.9715205068
3.095467	.8796313211	1.485824160	1.127023800	1.000082409

TABLE 1. Comparison table of conditions (3. 55), (3. 60) and (3. 61)

$$\| F'(x_0)^{-1} \| \leq \frac{8}{8-3\,|\zeta|}, \quad \eta = \frac{|\zeta|}{8-3\,|\zeta|}, \quad L = \frac{12\,|\zeta|}{8-3\,|\zeta|},$$

$$K = \frac{6\,r\,|\zeta|}{8-3\,|\zeta|} \quad and \quad h_\star = \frac{12\,r\,|\zeta|^2}{(8-3\,|\zeta|)^2}.$$

Denote by $\zeta^\star$ the positive root of equation $3\,(4\,r-3)\,t^2+48\,t-64=0$. Notice that if $\zeta > \zeta^\star$, then $h_\star > 1$. Hence the Newton-Kantorovich condition is not satisfied. Let us choose for example $r=3$. Then, we obtain $\mu^\star = .888889$.

In Table 1, we pick some values of r for $\zeta = 1$, so we give the corresponding values of $\zeta^\star$ and we compare the "h" conditions. We have chosen $K_0 = L_0 = L$. Hence, Table 1 shows that our conditions are always better than the Newton-Kantorovich conditions "$h_\star$" (see the third column of Table 1). **Local Convergence** Let $A(x)^\# = F'(x)^{-1}$, $x \in$, $\mathcal{X} == \mathbb{R}^3$, $= \overline{U}(0,1)$ and $= (0,0,0)$. Define functions F, G on for $w=(x,y,z)$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}\,y^2 + y, z) \tag{3. 63}$$

and

$$G(x) = \varepsilon(|x|,|y|,|z|) \text{ for some } \varepsilon \in (0,1).$$

Then, the Fréchet derivative of F is given by

$$F'(w) = \begin{pmatrix} e^x & 0 & 0 \\ 0 & (e-1)\,y+1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Notice that we have $F()=0$, $F'()=F'()^{-1}=\operatorname{diag}\{1,1,1\}$ and $L=e-1<K_0=K=e$, $M=\mu_0=\ell=\ell_0=\mu^*=\lambda=0$ and $\mu_1=\mu_2=\varepsilon$. Then, Theorem 2 gives that the convergence radius is

$$R = \frac{2(1-\varepsilon)}{3e-2},$$

whereas the convergence radius given by others [12, 13, 14, 27, 29, 30] (using $K=L$) is

$$R_0 = \frac{2(1-\varepsilon)}{3e}.$$

Notice that $R_0 < R$.

More examples where $L < K$ or $R_0 < R$ can be found in [5]-[11].

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