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Approximation of Solution of Time Fractional Order Three-Dimensional Heat Conduction Problems with Jacobi Polynomials

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Abstract. In this paper, we extend the idea of pseudo spectral method to approximate solution of time fractional order three-dimensional heat conduction equations on a cubic domain. We study shifted Jacobi polynomials and provide a simple scheme to approximate function of multi variables in terms of these polynomials. We develop new operational matrices for arbitrary order integrations as well as for arbitrary order derivatives. Based on these new matrices, we develop simple technique to obtain numerical solution of fractional order heat conduction equations. The new scheme is simple and can be easily simulated with any computational software. We develop codes for our results using MatLab. The results are displayed graphically.

AMS (MOS) Subject Classification Codes: 65N35; 65M70; 35C11
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1. INTRODUCTION

In literature, diffusion equations play basic and important role in mathematical modeling of large variety of engineering problems. Some of the important problems in which diffusion equation plays a basic role are: cyclic heating of the cylindrical surface of internal combustion engines, heating and cooling of building structures, heating lakes and water reservoirs by radiation, the heating of solid surfaces in materials processing, the cyclic heating of laminated steel during pickling, heating and cooling of vials contained DNA for polimerise-chain-reaction activation, the heating of electronics and many more see for example [12, 11, 1, 35, 3, 25]. In this study we consider the following generalized time fractional heat conduction problem

$$\begin{aligned}\chi_t \frac{\partial^\sigma u(t,x,y,z)}{\partial t^\sigma} &= \lambda_x \frac{\partial^2 u(t,x,y,z)}{\partial x^2} + \lambda_y \frac{\partial^2 u(t,x,y,z)}{\partial y^2} + \\ \lambda_z \frac{\partial^2 u(t,x,y,z)}{\partial z^2} &+ I(t,x,y,z), \qquad u(0,x,y,z) = f(x,y,z),\end{aligned} \tag{1. 1}$$

where χ_t is the volumetric heat capacity and $\lambda_x, \lambda_y \, and \, \lambda_z$ are the thermal conductivities in the x, y and z directions, $0 < \sigma \leq 1$ is the order of the derivative, $t \in [0, \tau]$, $x \in [0, a]$, $y \in [0, b]$ and $z \in [0, c]$. $I(t, x, y, z)$ is the internal source term and $f(x, y, z)$ is the initial heat distribution in the space.

Exact analytical solution of time fractional order diffusion equations are generally very difficult to obtain. The reason behind this difficulty is the higher computational complexities of fractional calculus involved in solving diffusion equations. This phenomena is recently been reported by many authors. We refer to Poulikakos [28], Arpaci [4], O zisik [26], Kakac and Yener [15], and Carslaw and Jaeger [5] for some of the renowned results in this aspect. Different aspects of solution of the problem such as existence and uniqueness of positive solutions, analytical properties of solution and the correctness of initial and boundary conditions have already been studied by many authors. We refer to study [38, 34, 22, 23, 13, 14, 9, 21].

In the literature, many attempts were made to approximate solution of fractional diffusion equations. V. V. Kulish [20] provide a very efficient way for approximate solution of such problems using laplace transform. M. Akbarzade [2] studied approximate solution of integer order three dimensional transient state heat conduction equation by Homotopy analysis method. However, for fractional order equations, the method used in [2] will results very complex algorithms. Ting-Hui Ning [24] provided some results based on spherical coordinates and the method of separation of variables for approximate solution of such type of problems. Y. Z. Povstenko [29] provided axisymmetric solutions to time-fractional heat conduction equation in a half-space subject to Robin boundary conditions by the use of integral transform method. Recently G.C. Wu [36, 37] studied approximate solution to fractional diffusion equation by variational iteration technique.

One of the most powerful method for numerical solutions of differential equations is the well known spectral method. This method has already been extensively used for numerical solutions of fractional order differential equations and partial differential equations with different types of boundary conditions, see for example [32, 27, 30, 31]. However, no generalized version of this method is available in the literature which can be used to deal with higher dimensional problems.

We provide generalized version of the method to find numerical solutions of higher dimensional fractional order partial differential equations. The methods is based on operational matrices of integrations and differentiations. Operational matrices in case of single variable are available for different orthogonal polynomials such as Sine-Cosine, Legendre, Jacobi, Lagurree and hermite polynomials, we refer to [33, 6, 7, 10, 8, 16, 17, 18].

In this paper, we use two parametric shifted Jacobi polynomials and generalize the operational matrices of fractional order integrations and differentiations. We use these operational matrices to reduce the differential equation under consideration to a system of easily solvable algebraic equations.

The rest of the paper is organized as follow, in section 2 we provide some basic properties of fractional calculus and orthogonal polynomials and some relations for approximation of multivariate function. In section 3, we develop new operational matrices of integrations and differentiations, in section 4 these operational matrices are used to convert the corresponding differential equation to a system of algebraic equations. In section 5 the proposed algorithms are applied to several test problems and finally in section 6 a short conclusion is made.

2. PRELIMINARIES

For convenience, this section summarizes some concepts, definitions and basic results from fractional calculus in the sequel.

DEFINITION 1. *Given an interval* $[0, a] \subset \mathbb{R}$*, the Riemann-Liouville fractional order integral of order* $\sigma \in \mathbb{R}_+$ *of a function* $\phi \in (L^1[0, a], \mathbb{R})$ *is defined by*

$$\mathcal{I}_{0+}^{\sigma}\phi(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1}\phi(s)ds,$$

provided that the integral on right hand side exists.

DEFINITION 2. *For a given function* $\phi(t) \in C^n[0, a]$*, the Caputo fractional order derivative of order* σ *is defined as*

$$D^{\sigma}\phi(t) = \frac{1}{\Gamma(n-\sigma)} \int_0^x \frac{\phi^{(n)}(t)}{(x-t)^{\sigma+1-n}}dt,\ n-1 \le \sigma < n\ ,\ n \in N,$$

provided that the right side is pointwise defined on $(0, \infty)$*, where* $n = [\sigma] + 1$.

Hence, it follows that

$$D^{\sigma}t^k = \frac{\Gamma(1+k)}{\Gamma(1+k-\sigma)}t^{k-\sigma},\ I^{\sigma}t^k = \frac{\Gamma(1+k)}{\Gamma(1+k+\sigma)}t^{k+\sigma} \text{ and } D^{\sigma}C = 0, \text{ for a constant } C. \tag{2. 2}$$

2.1. **The shifted Jacobi polynomials:** [19] The well known two parametric Jacobi polynomials defined on $[0, \tau]$ with parameter ξ, ζ is given by the following relation

$$J_{(\tau,i)}^{(\xi,\zeta)}(t) = \sum_{l=0}^{i} \mho_{(l,i)}^{\tau} t^l, i = 0, 1, 2, 3..., \tag{2. 3}$$

where $\mho_{(l,i)}^{\tau}$ is defined by

$$\mho_{(l,i)}^{\tau} = \frac{(-1)^{i-l}\Gamma(i+\zeta+1)\Gamma(i+l+\xi+\zeta+1)}{\Gamma(l+\zeta+1)\Gamma(i+\xi+\zeta+1)(i-l)!l!\tau^l}. \tag{2. 4}$$

These polynomials are orthogonal with respect to the weight function

$$\omega_{\tau}^{(\xi,\zeta)}(t) = (\tau-t)^{\xi}t^{\zeta}.$$

The orthogonality condition of these polynomials are given as under

$$\int_0^\tau \omega_\tau^{(\xi,\zeta)}(t) J_{(\tau,i)}^{(\xi,\zeta)}(t) J_{(\tau,j)}^{(\xi,\zeta)}(t) = \gamma_{(\tau,j)}^{(\xi,\zeta)} \delta_{i,j}, \tag{2. 5}$$

where $\delta_{i,j}$ is the Kroncker delta and $\gamma_{(\tau,j)}^{(\xi,\zeta)}$ is defined as

$$\gamma_{(\tau,j)}^{(\xi,\zeta)} = \frac{\tau^{\xi+\zeta+1}\Gamma(j+\xi+1)\Gamma(j+\zeta+1)}{(2j+\xi+\zeta+1)\Gamma(j+1)\Gamma(j+\xi+\zeta+1)}. \tag{2. 6}$$

The orthogonality relation allows us to approximate $u(t) \in C([0,\tau]$ in the form of Jacobi series as follows

$$u(t) = \sum_{i=0}^{\infty} c_i J_{(\tau,i)}^{(\xi,\zeta)}(t), \tag{2. 7}$$

where c_i can be easily calculated by using the orthogonality relation, that is,

$$c_i = \frac{1}{\gamma_{(\tau,j)}^{(\xi,\zeta)}} \int_0^\tau u(t) \omega_\tau^{(\xi,\zeta)}(t) J_{(\tau,i)}^{(\xi,\zeta)}(t) dt.$$

It is clear from Lemma 2.2.1 in [19] that the coefficients c_i decay faster. In practice, we are concerned with the truncated series of (2. 7). The m terms truncated series can be written in vector form as

$$u(t) \simeq K_M^T \Lambda_M(t), \tag{2. 8}$$

where $M = m+1$, K_M is the coefficient column vector and $\Lambda_M(t)$ is M terms column vector function defined by

$$\Lambda_M(t) = \begin{bmatrix} J_{(\tau,0)}^{(\xi,\zeta)}(t) & J_{(\tau,1)}^{(\xi,\zeta)}(t) & \cdots & J_{(\tau,i)}^{(\xi,\zeta)}(t) & \cdots & J_{(\tau,m)}^{(\xi,\zeta)}(t) \end{bmatrix}^T. \tag{2. 9}$$

2.2. **Error Estimate.** For sufficiently smooth function $u(t) \in \Delta$, where $\Delta = [a,b]$, the maximum amount of error in the approximation of a function with m terms of Jacobi polynomials is given as

$$\|g(x) - g_{(M)}(x)\|_2 \leq (C_1 \frac{1}{M^{M+1}}), \tag{2. 10}$$

where

$$C_1 = \frac{1}{4} \max_{t\in[0,\tau]} |\frac{\partial^{M+1}}{\partial t^{M+1}} u(x)|. \tag{2. 11}$$

For the proof of this relation, we refer the reader to study [19]. In our current paper, we are concerned with four dimensional problems. Therefore, we must first establish a suitable approximation method of a function of four variable with Jacobi polynomials. Using the same procedure as developed in [16], we extend the notion to three-dimensional space and define three-dimensional Jacobi polynomials of order M on the domain $[0,a] \times [0,b] \times [0,c]$ as a product function of three Jacobi polynomials

$$J_{(q,r,s)}^{(a,b,c)} = J_{(a,q)}^{(\xi,\zeta)}(x) J_{(b,r)}^{(\xi,\zeta)}(y) J_{(c,s)}^{(\xi,\zeta)}(z). \tag{2. 12}$$

The orthogonality condition of $J_{(q,r,s)}^{(a,b,c)}$ is found to be

$$\int_0^c \int_0^b \int_0^a J_{(q,r,s)}^{(a,b,c)} J_{(q',r',s')}^{(a,b,c)} \omega_{a,b,c}^{(\xi,\zeta)} dxdydz$$
$$= \delta_{(q,q')} \delta_{(r,r')} \delta_{(s,s')} \gamma_{(a,q)}^{(\xi,\zeta)} \gamma_{(b,r)}^{(\xi,\zeta)} \gamma_{(c,s)}^{(\xi,\zeta)},$$

where $\omega_{a,b,c}^{(\xi,\zeta)} = \omega_a^{(\xi,\zeta)}(x)\omega_b^{(\xi,\zeta)}(y)\omega_c^{(\xi,\zeta)}(z)$ is the weight function regarding three dimensional Jacobi polynomials. Hence, any $u(x,y,z) \in C([0,a]\times[0,b]\times[0,c])$ can be easily approximated with three dimensional Jacobi polynomials $J_{(q,r,s)}^{(a,b,c)}$ as follows

$$u(x,y,z) = \sum_{q=0}^{m}\sum_{r=0}^{m}\sum_{s=0}^{m} c_{qrs} J_{(q,r,s)}^{(a,b,c)}, \tag{2. 13}$$

where c_{qrs} can be obtained by using the following relation

$$c_{qrs} = \frac{1}{\gamma_{(a,q)}^{(\xi,\zeta)}\gamma_{(b,r)}^{(\xi,\zeta)}\gamma_{(c,s)}^{(\xi,\zeta)}} \int_0^c\int_0^b\int_0^a u(x,y,z)\omega_{a,b,c}^{(\xi,\zeta)} J_{(q,r,s)}^{(a,b,c)} dxdydz. \tag{2. 14}$$

For simplicity, use the notation $c_n = c_{qrs}$ where $n = M^2q + Mr + s + 1$, and rewrite (2. 13) in vector notation, as follows

$$u(x,y,z) = \sum_{n=1}^{M^3} c_n J_{(n)}^{(a,b,c)}(x,y,z) = C_{M^3}^T \Lambda^{(a,b,c)}(x,y,z).$$

Where C_{M^3} is coefficient column vector of order M^3 and $\Lambda^{(a,b,c)}(x,y,z)$ is column vector of functions defined by

$$\Lambda^{(a,b,c)}(x,y,z) = \begin{bmatrix} J_{(1)}^{(a,b,c)} & J_{(2)}^{(a,b,c)} & \cdots J_{(n)}^{(a,b,c)} & \cdots & J_{(M^3)}^{(a,b,c)} \end{bmatrix}^T. \tag{2. 15}$$

2.3. **Four-dimensional Jacobi polynomials.** Now, extend the idea to four dimensional space defined on domain $[0,\tau]\times[0,a]\times[0,b]\times[0,c]$ by the product function of Jacobi polynomials of order M as

$$J_{(p,q,r,s)}^{(\tau,a,b,c)} = J_{(\tau,p)}^{(\xi,\zeta)}(t) J_{(a,q)}^{(\xi,\zeta)}(x) J_{(b,r)}^{(\xi,\zeta)}(y) J_{(c,s)}^{(\xi,\zeta)}(z). \tag{2. 16}$$

The orthogonality condition of $J_{(p,q,r,s)}^{(\tau,a,b,c)}$ is found to be

$$\int_0^c\int_0^b\int_0^a\int_0^\tau J_{(p,q,r,s)}^{(\tau,a,b,c)} J_{(p',q',r',s')}^{(\tau,a,b,c)} \omega_{\tau,a,b,c}^{(\xi,\zeta)} dtdxdydz$$
$$= \delta_{(p,p')}\delta_{(q,q')}\delta_{(r,r')}\delta_{(s,s')}\gamma_{(\tau,p)}^{(\xi,\zeta)}\gamma_{(a,q)}^{(\xi,\zeta)}\gamma_{(b,r)}^{(\xi,\zeta)}\gamma_{(c,s)}^{(\xi,\zeta)},$$

where $\omega_{\tau,a,b,c}^{(\xi,\zeta)} = \omega_\tau^{(\xi,\zeta)}(t)\omega_a^{(\xi,\zeta)}(x)\omega_b^{(\xi,\zeta)}(y)\omega_c^{(\xi,\zeta)}(z)$ is the weight function regarding four-dimensional Jacobi polynomials. Any $u(t,x,y,z) \in C([0,\tau]\times[0,a]\times[0,b]\times[0,c])$ can be easily approximated with four dimensional Jacobi polynomials $J_{(p,q,r,s)}^{(\tau,a,b,c)}$ as follows

$$u(t,x,y,z) = \sum_{p=0}^{m}\sum_{q=0}^{m}\sum_{r=0}^{m}\sum_{s=0}^{m} d_{pqrs} J_{(p,q,r,s)}^{(\tau,a,b,c)}, \tag{2. 17}$$

where d_{pqrs} can be obtained by the relation

$$d_{pqrs} = \frac{1}{\gamma_{(\tau,p)}^{(\xi,\zeta)}\gamma_{(a,q)}^{(\xi,\zeta)}\gamma_{(b,r)}^{(\xi,\zeta)}\gamma_{(c,s)}^{(\xi,\zeta)}} \int_0^c\int_0^b\int_0^a\int_0^\tau u(t,x,y,z)\omega_{\tau,a,b,c}^{(\xi,\zeta)} J_{(p,q,r,s)}^{(\tau,a,b,c)} dtdxdydz.$$

For simplicity, we use the notation $d_{p,n} = d_{pqrs}$ where $n = M^2q + Mr + s + 1$, and rewrite (2. 17) as follows

$$\begin{aligned} u(t,x,y,z) &= \sum_{p=0}^{m}\sum_{n=1}^{M^3} d_{p,n} J_p^{\tau}(t) J_n^{(a,b,c)}(x,y,z) \\ &= \Lambda_M^T(t) D_{M\times M^3} \Lambda^{(a,b,c)}(x,y,z), \end{aligned}$$

where $\Lambda_M^T(t)$ is the function vector related to variable t and is as defined in (2. 9) and $\Lambda^{(a,b,c)}(x,y,z)$ is the function vector related to variable x, y, z and is defined in (2. 15).

3. Operational matrices of Integration and Differentiations

The operational matrices of derivatives and integrations play the role of building blocks in the establishment of the new pseudo spectral method. In literature, these are used only for solutions of differential equations including fractional differential with only one variable and partial differential equations with two and three variables [16]. Here we construct new operational matrices for three variables and use them to convert a generalized class of PDEs with four variable to a system of easily solvable algebraic equations.

Lemma 3.1. *Let* $\Lambda^{(a,b,c)}(x,y,z)$ *be the function vector as defined in* (2. 15)*, then the fractional order partial derivative of order* σ *of* $\Lambda^{(a,b,c)}(x,y,z)$ *w.r.t* x *is given by*

$$\frac{\partial^{\sigma}}{\partial x^{\sigma}} \Lambda^{(a,b,c)}(x,y,z) = {}^{x}A_{M^3\times M^3}^{(\sigma,a,b,c)} \Lambda^{(a,b,c)}(x,y,z),$$

where ${}^{x}A_{M^3\times M^3}^{(\sigma,a,b,c)}$ *is the operational matrix of differentiation of order* σ*, and is defined as*

$${}^{x}A_{M^3\times M^3}^{(\sigma,a,b,c)} = \begin{bmatrix} \Omega_{1,1} & \Omega_{1,2} & \cdots & \Omega_{1,n'} & \cdots & \Omega_{1,M^3} \\ \Omega_{2,1} & \Omega_{2,2} & \cdots & \Omega_{2,n'} & \cdots & \Omega_{2,M^3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Omega_{n,1} & \Omega_{n,2} & \cdots & \Omega_{n,n'} & \cdots & \Omega_{n,M^3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Omega_{M^3,1} & \Omega_{M^3,2} & \cdots & \Omega_{M^3,n'} & \cdots & \Omega_{M^3,M^3} \end{bmatrix}, \qquad (3.\ 18)$$

where

$$n' = M^2h + Mi + j + 1,\ n = M^2q + Mr + s + 1,$$
$$i, j, k, q, r, s = 0, 1, 2, ..., m,$$

and

$$\Omega_{n,n'} = \widehat{h}_{n'}^{n} = \widehat{h}_{ijk}^{qrs} = \sum_{l=\lceil\sigma\rceil}^{q} h_{ijk}^{qrs} \overbrace{\mho_{(l,q)}^{a}}, \qquad (3.\ 19)$$

$$h_{ijk}^{qrs} = \frac{\delta_{(j,r)}\delta_{(k,s)}}{\gamma_{(a,i)}^{(\xi,\zeta)}} \sum_{l'=0}^{i} \mho_{(l',i)}^{a} \frac{\Gamma(l'+\zeta+l-\sigma+1)\Gamma(\xi+1)a^{(l'+\zeta+l-\sigma+\xi+1)}}{\Gamma(l'+\zeta+l-\sigma+\xi+1)},$$

$\overbrace{\mho_{(l,q)}^{a}} = \mho_{(l,q)}^{a} \frac{1+l}{1+l-\sigma}$, *and* $\mho_{(.,.)}^{a}$ *is as defined in* (2. 4).

Proof. Consider the general term of the function vector (2. 15). Then, in view of (2. 12), we can write

$$\frac{\partial^{\sigma}}{\partial x^{\sigma}} J_{(n)}^{(a,b,c)}(x,y,z) = \frac{\partial^{\sigma}}{\partial x^{\sigma}} J_{(q,r,s)}^{(a,b,c)}, \qquad (3.\ 20)$$

where $n = M^2q + Mr + s + 1$. After expansion of the left side we get

$$\frac{\partial^\sigma}{\partial x^\sigma} J^{(a,b,c)}_{(n)}(x,y,z) = J^{(\xi,\zeta)}_{(b,r)}(y) J^{(\xi,\zeta)}_{(c,s)}(z) \frac{\partial^\sigma}{\partial x^\sigma} J^{(\xi,\zeta)}_{(a,q)}(x). \tag{3. 21}$$

Using (2. 2) and (2. 3), and after simplification, we obtain

$$\frac{\partial^\sigma}{\partial x^\sigma} J^{(a,b,c)}_{(n)}(x,y,z) = J^{(\xi,\zeta)}_{(b,r)}(y) J^{(\xi,\zeta)}_{(c,s)}(z) \sum_{l=\lceil\sigma\rceil}^{q} \mho^a_{(l,q)} \frac{1+l}{1+l-\sigma} x^{l-\sigma}, \tag{3. 22}$$

which can be written in the following form

$$\frac{\partial^\sigma}{\partial x^\sigma} J^{(a,b,c)}_{(n)}(x,y,z) = \sum_{l=\lceil\sigma\rceil}^{q} \widehat{\mho^a_{(l,q)}} J^{(\xi,\zeta)}_{(b,r)}(y) J^{(\xi,\zeta)}_{(c,s)}(z) x^{l-\sigma}, \tag{3. 23}$$

where $\widehat{\mho^a_{(l,q)}} = \mho^a_{(l,q)} \frac{1+l}{1+l-\sigma}$. Approximating $J^{(\xi,\zeta)}_{(b,r)}(y) J^{(\xi,\zeta)}_{(c,s)}(z) x^{l-\sigma}$ with three dimensional Jacobi polynomials as follows

$$J^{(\xi,\zeta)}_{(b,r)}(y) J^{(\xi,\zeta)}_{(c,s)}(z) x^{l-\sigma} = \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{m} h^{qrs}_{ijk} J^{(a,b,c)}_{(i,j,k)}. \tag{3. 24}$$

The coefficients h^{qrs}_{ijk} can be easily calculated by using relation (2. 14), that is,

$$h^{qrs}_{ijk} = \frac{1}{\gamma^{(\xi,\zeta)}_{(a,i)}\gamma^{(\xi,\zeta)}_{(b,j)}\gamma^{(\xi,\zeta)}_{(c,k)}} \int_0^c \int_0^b \int_0^a \omega^{(\xi,\zeta)}_{a,b,c} J^{(\xi,\zeta)}_{(b,r)}(y) J^{(\xi,\zeta)}_{(c,s)}(z) x^{l-\sigma} J^{(a,b,c)}_{(i,j,k)} dxdydz,$$

which after simplification yields

$$h^{qrs}_{ijk} = \frac{\delta_{(j,r)}\delta_{(k,s)}}{\gamma^{(\xi,\zeta)}_{(a,i)}} \int_0^a \omega^{(\xi,\zeta)}_a x^{l-\sigma} J^{(\xi,\zeta)}_{(a,i)}(x)dx. \tag{3. 25}$$

Now, we obtain

$$\int_0^a \omega^{(\xi,\zeta)}_a x^{l-\sigma} J^{(\xi,\zeta)}_{(a,i)}(x)dx = \sum_{l'=0}^{i} \mho^a_{(l',i)} \int_0^a (a-x)^\xi x^{l'+\zeta+l-\sigma}dx. \tag{3. 26}$$

By the convolution theorem of Laplace transform, we have

$$\pounds\left(\int_0^a (a-x)^\xi x^{l'+\zeta+l-\sigma}dx\right) = \frac{\Gamma(l'+\zeta+l-\sigma+1)\Gamma(\xi+1)}{s^{(l'+\zeta+l-\sigma+\xi+2)}},$$

and by taking the inverse Laplace transform, we obtain

$$\int_0^a (a-x)^\xi x^{l'+\zeta+l-\sigma}dx = \frac{\Gamma(l'+\zeta+l-\sigma+1)\Gamma(\xi+1)a^{(l'+\zeta+l-\sigma+\xi+1)}}{\Gamma(l'+\zeta+l-\sigma+\xi+1)}. \tag{3. 27}$$

Using the equation (3. 27) and (3. 26) in (3. 25), we get

$$h^{qrs}_{ijk} = \frac{\delta_{(j,r)}\delta_{(k,s)}}{\gamma^{(\xi,\zeta)}_{(a,i)}} \sum_{l'=0}^{i} \mho^a_{(l',i)} \frac{\Gamma(l'+\zeta+l-\sigma+1)\Gamma(\xi+1)a^{(l'+\zeta+l-\sigma+\xi+1)}}{\Gamma(l'+\zeta+l-\sigma+\xi+1)}. \tag{3. 28}$$

Also by using (3. 24) in (3. 23) we get

$$\begin{aligned}\frac{\partial^\sigma}{\partial x^\sigma} J^{(a,b,c)}_{(n)}(x,y,z) &= \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{m}\sum_{l=\lceil\sigma\rceil}^{q} h^{qrs}_{ijk}\,\widehat{\mho^{a}_{(l,q)}}\, J^{(a,b,c)}_{(i,j,k)} \\ &= \sum_{i=0}^{m}\sum_{j=0}^{m}\sum_{k=0}^{m} \widehat{h}^{qrs}_{ijk} J^{(a,b,c)}_{(i,j,k)},\end{aligned} \tag{3. 29}$$

where $\widehat{h}^{qrs}_{ijk} = \sum_{l=\lceil\sigma\rceil}^{q} h^{qrs}_{ijk}\widehat{\mho^{a}_{(l,q)}}$. Now using the notation $n = M^2q + Mr + s + 1$, $n' = M^2 i + Mj + k + 1$, we have

$$\frac{\partial^\sigma}{\partial x^\sigma} J^{(a,b,c)}_{(n)}(x,y,z) = \sum_{n'=0}^{M^3} \widehat{h}^{n}_{n'} J^{(a,b,c)}_{(n')}(x,y,z). \tag{3. 30}$$

which in view of the notation $\Omega_{n,n'} = \widehat{h}^{n}_{n'}$ for $i,j,k,q,r,s = 0,1,2,3,..m$, yields the desired result. □

LEMMA 3.2. *Let $\Lambda^{(a,b,c)}(x,y,z)$ be the function vector as defined in* (2. 15)*, then the fractional order partial derivative of order σ of $\Lambda^{(a,b,c)}(x,y,z)$ w.r.t y is given by*

$$\frac{\partial^\sigma}{\partial y^\sigma} \Lambda^{(a,b,c)}(x,y,z) = {}^{y}A^{(\sigma,a,b,c)}_{M^3\times M^3} \Lambda^{(a,b,c)}(x,y,z), \tag{3. 31}$$

where ${}^{y}A^{(\sigma,a,b,c)}_{M^3\times M^3}$ is the operational matrix of differentiation of order σ, and is given by

$${}^{y}A^{(\sigma,a,b,c)}_{M^3\times M^3} = \begin{bmatrix} \Omega_{1,1} & \Omega_{1,2} & \cdots & \Omega_{1,n'} & \cdots & \Omega_{1,M^3} \\ \Omega_{2,1} & \Omega_{2,2} & \cdots & \Omega_{2,n'} & \cdots & \Omega_{2,M^3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Omega_{n,1} & \Omega_{n,2} & \cdots & \Omega_{n,n'} & \cdots & \Omega_{n,M^3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Omega_{M^3,1} & \Omega_{M^3,2} & \cdots & \Omega_{M^3,n'} & \cdots & \Omega_{M^3,M^3} \end{bmatrix}, \tag{3. 32}$$

where

$$n' = M^2h + Mi + j + 1,\; n = M^2q + Mr + s + 1,$$
$$i,j,k,q,r,s = 0,1,2,...,m,$$

and

$$\Omega_{n,n'} = \widehat{h}^{n}_{n'} = \widehat{h}^{qrs}_{ijk} = \sum_{l=\lceil\sigma\rceil}^{r} h^{qrs}_{ijk}\widehat{\mho^{b}_{(l,r)}}, \tag{3. 33}$$

$$h^{qrs}_{ijk} = \frac{\delta_{(i,q)}\delta_{(k,s)}}{\gamma^{(\xi,\zeta)}_{(b,j)}} \sum_{l'=0}^{j} \mho^{b}_{(l',j)} \frac{\Gamma(l'+\zeta+l-\sigma+1)\Gamma(\xi+1)b^{(l'+\zeta+l-\sigma+\xi+1)}}{\Gamma(l'+\zeta+l-\sigma+\xi+1)},$$

$\widehat{\mho^{b}_{(l,r)}} = \mho^{b}_{(l,r)}\frac{1+l}{1+l-\sigma}$*, and $\mho^{b}_{(.,.)}$ is as defined in* (2. 4)*.*

Proof. The proof of this lemma is similar to that of the above lemma. □

LEMMA 3.3. *Let $\Lambda^{(a,b,c)}(x,y,z)$ be the function vector as defined in* (2. 15)*, then the fractional order partial derivative of order σ of $\Lambda^{(a,b,c)}(x,y,z)$ w.r.t z is given by*

$$\frac{\partial^\sigma}{\partial z^\sigma} \Lambda^{(a,b,c)}(x,y,z) = {}^{z}A^{(\sigma,a,b,c)}_{M^3\times M^3} \Lambda^{(a,b,c)}(x,y,z), \tag{3. 34}$$

where ${}^{z}A^{(\sigma,a,b,c)}_{M^3\times M^3}$ *is the operational matrix of differentiation of order* σ*, and is defined by*

$$
{}^{z}A^{(\sigma,a,b,c)}_{M^3\times M^3} = \begin{bmatrix} \Omega_{1,1} & \Omega_{1,2} & \cdots & \Omega_{1,n'} & \cdots & \Omega_{1,M^3} \\ \Omega_{2,1} & \Omega_{2,2} & \cdots & \Omega_{2,n'} & \cdots & \Omega_{2,M^3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Omega_{n,1} & \Omega_{n,2} & \cdots & \Omega_{n,n'} & \cdots & \Omega_{n,M^3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Omega_{M^3,1} & \Omega_{M^3,2} & \cdots & \Omega_{M^3,n'} & \cdots & \Omega_{M^3,M^3} \end{bmatrix}, \tag{3. 35}
$$

where

$$
n' = M^2h + Mi + j + 1,\; n = M^2q + Mr + s + 1,
$$

$$
i, j, k, q, r, s = 0, 1, 2, ..., m,
$$

and

$$
\Omega_{n,n'} = \widehat{h}^{n}_{n'} = \widehat{h}^{qrs}_{ijk} = \sum_{l=\lceil\sigma\rceil}^{s} h^{qrs}_{ijk}\,\widehat{\mho^{c}_{(l,s)}}, \tag{3. 36}
$$

$$
h^{qrs}_{ijk} = \frac{\delta_{(i,q)}\delta_{(j,r)}}{\gamma^{(\xi,\zeta)}_{(c,k)}} \sum_{l'=0}^{k} \mho^{c}_{(l',k)} \frac{\Gamma(l'+\zeta+l-\sigma+1)\Gamma(\xi+1)c^{(l'+\zeta+l-\sigma+\xi+1)}}{\Gamma(l'+\zeta+l-\sigma+\xi+1)},
$$

$\widehat{\mho^{c}_{(l,s)}} = \mho^{c}_{(l,s)}\frac{1+l}{1+l-\sigma}$ *and* $\mho^{c}_{(.,.)}$ *is as defined in* (2. 4).

Proof. The proof of this lemma is similar to that of the above lemma. □

LEMMA 3.4. *Let* $\Lambda_M(t)$ *be the function vector as defined in* (2. 9) *then the* γ *order integration of* $\Lambda_M(t)$ *is given by*

$$
I^{\gamma}(\Lambda_M(t)) \simeq H^{\eta,\gamma}_{M\times M}\Lambda_M(t), \tag{3. 37}
$$

where $H^{\eta,\gamma}_{M\times M}$ *is the operational matrix of integration of order* γ *and is defined as*

$$
H^{\eta,\gamma}_{M\times M} = \begin{bmatrix} \Theta_{0,0,k} & \Theta_{0,1,k,} & \cdots & \Theta_{0,j,k} & \cdots & \Theta_{0,m,k} \\ \Theta_{1,0,k} & \Theta_{1,1,k} & \cdots & \Theta_{1,j,k} & \cdots & \Theta_{1,m,j} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Theta_{i,0,k} & \Theta_{i,1,k} & \cdots & \Theta_{i,j,k} & \cdots & \Theta_{i,m,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Theta_{m,0,k} & \Theta_{m,1,k} & \cdots & \Theta_{m,j,k} & \cdots & \Theta_{m,m,k} \end{bmatrix} \tag{3. 38}
$$

where

$$
\Theta_{i,j,k} = \sum_{k=0}^{i} \Lambda_{i,k,\gamma} S_j, \tag{3. 39}
$$

$$
\Lambda_{i,k,\gamma} = \frac{(-1)^{i-k}\Gamma(i+\zeta+1)\Gamma(i+k+\xi+\zeta+1)\Gamma(1+k)}{\Gamma(k+\zeta+1)\Gamma(i+\xi+\zeta+1)(i-k)!k!\Gamma(1+k+\xi)\eta^{k}}, \tag{3. 40}
$$

and

$$S_j = \sum_{l=0}^{j} \frac{(-1)^{j-l}\Gamma(j+l+\xi+\zeta+1)\Gamma(k+\xi+l+\zeta+1)}{\Gamma(l+\zeta+1)(j-l)!l!\Gamma(k+\gamma+l+\zeta+\xi+2)} \times \frac{(2j+\xi+\zeta+1)\Gamma(j+1)\Gamma(\xi+1)\eta^{\gamma}}{\Gamma(j+\xi+1)}.$$

Proof. For the proof of this lemma we refer to [19]. □

4. Application of the operational matrices of integrations and derivatives to heat conduction problem

Consider the following heat conduction problem posed on a cubic region

$$\chi_t \frac{\partial^{\sigma} u(t,x,y,z)}{\partial t^{\sigma}} = \lambda_x \frac{\partial^2 u(t,x,y,z)}{\partial x^2} + \lambda_y \frac{\partial^2 u(t,x,y,z)}{\partial y^2} + \lambda_z \frac{\partial^2 u(t,x,y,z)}{\partial z^2} + I(t,x,y,z), \qquad u(0,x,y,z) = f(x,y,z), \tag{4. 41}$$

where χ_t is the volumetric heat capacity $(J/(m^3K))$, $\lambda_x, \lambda_y, \lambda_z$ are thermal conductivities $(W/m.K)$ in x, y and z direction respectively, $0 < \alpha \leq 1$, $t \in [0,\tau]$, $x \in [0,a]$, $y \in [0,b]$ and $z \in [0,c]$. We seek solution of this problem in terms of shifted Jacobi polynomials such that the following relation holds

$$\frac{\partial^{\sigma} u(t,x,y,z)}{\partial t^{\sigma}} = \Lambda_M(t)^T K_{M\times M^3} \Lambda^{(a,b,c)}(x,y,z). \tag{4. 42}$$

By the application of fractional integration of order σ with respect to variable t on equation (4. 42) and making use of Lemma 3.4, we have

$$I^{\sigma} \frac{\partial^{\sigma} u(t,x,y,z)}{\partial t^{\sigma}} = \Lambda_M(t)^T (H^{\tau,\sigma}_{M\times M})^T K_{M\times M^3} \Lambda^{(a,b,c)}(x,y,z),$$

which implies that

$$u(t,x,y,z) - c_1 = \Lambda_M(t)^T (H^{\tau,\sigma}_{M\times M})^T K_{M\times M^3} \Lambda^{(a,b,c)}(x,y,z). \tag{4. 43}$$

The initial condition yields $c_1 = u(0,x,y,z)$. Hence, we have

$$u(t,x,y,z) = \Lambda_M(t)^T (H^{\tau,\sigma}_{M\times M})^T K_{M\times M^3} \Lambda^{(a,b,c)}(x,y,z) + f(x,y,z), \tag{4. 44}$$

which can be rewritten as

$$u(t,x,y,z) = \Lambda_M(t)^T \{(H^{\tau,\sigma}_{M\times M})^T K_{M\times M^3} + F_{M\times M^3}\} \Lambda^{(a,b,c)}(x,y,z). \tag{4. 45}$$

Using this value of $u(t,x,y,z)$, we obtain

$$\frac{\partial^2 u(t,x,y,z)}{\partial x^2} = \Lambda_M(t)^T \{(H^{\tau,\sigma}_{M\times M})^T K_{M\times M^3} + F_{M\times M^3}\}\, {}^{x}A^{(\sigma,a,b,c)}_{M^3\times M^3} \Lambda^{(a,b,c)}(x,y,z), \tag{4. 46}$$

$$\frac{\partial^2 u(t,x,y,z)}{\partial y^2} = \Lambda_M(t)^T \{(H^{\tau,\sigma}_{M\times M})^T K_{M\times M^3} + F_{M\times M^3}\}\, {}^{y}A^{(\sigma,a,b,c)}_{M^3\times M^3} \Lambda^{(a,b,c)}(x,y,z) \tag{4. 47}$$

and

$$\frac{\partial^2 u(t,x,y,z)}{\partial z^2} = \Lambda_M(t)^T \{(H^{\tau,\sigma}_{M\times M})^T K_{M\times M^3} + F_{M\times M^3}\}\, {}^{z}A^{(\sigma,a,b,c)}_{M^3\times M^3} \Lambda^{(a,b,c)}(x,y,z). \tag{4. 48}$$

Using (4. 42), (4. 46), (4. 47) and (4. 48) in (4. 41), we get

$$\chi_t\Lambda_M(t)^TK_{M\times M^3}\Lambda^{(a,b,c)}(x,y,z)) = \lambda_x\Lambda_M(t)^T\{(H^{\tau,\sigma}_{M\times M})^TK_{M\times M^3}+F_{M\times M^3}\}\,{}^xA^{(\sigma,a,b,c)}_{M^3\times M^3}$$
$$+\lambda_y\Lambda_M(t)^T\{(H^{\tau,\sigma}_{M\times M})^TK_{M\times M^3}+F_{M\times M^3}\}\,{}^yA^{(\sigma,a,b,c)}_{M^3\times M^3}\Lambda^{(a,b,c)}(x,y,z)$$
$$+\lambda_z\Lambda_M(t)^T\{(H^{\tau,\sigma}_{M\times M})^TK_{M\times M^3}+F_{M\times M^3}\}\,{}^zA^{(\sigma,a,b,c)}_{M^3\times M^3}\Lambda^{(a,b,c)}(x,y,z)$$
$$+\Lambda_M(t)^TF^o_{M\times M^3}\Lambda^{(a,b,c)}(x,y,z), \tag{4. 49}$$

where $\Lambda_M(t)^TF^o_{M\times M^3}\Lambda^{(a,b,c)}(x,y,z) = I(t,x,y,z)$. The above equations can be simplified as

$$\Lambda_M(t)^T[(\chi_tK_{M\times M^3}-\lambda_x\{(H^{\tau,\sigma}_{M\times M})^TK_{M\times M^3}+F_{M\times M^3}\}\,{}^xA^{(\sigma,a,b,c)}_{M^3\times M^3}-$$
$$\lambda_y\{(H^{\tau,\sigma}_{M\times M})^TK_{M\times M^3}+F_{M\times M^3}\}\,{}^yA^{(\sigma,a,b,c)}_{M^3\times M^3}-$$
$$\lambda_z\{(H^{\tau,\sigma}_{M\times M})^TK_{M\times M^3}+F_{M\times M^3}\}\,{}^zA^{(\sigma,a,b,c)}_{M^3\times M^3}-F^o_{M\times M^3}]\Lambda^{(a,b,c)}(x,y,z)=0.$$

which implies that

$$[(\chi_tK_{M\times M^3}-\lambda_x\{(H^{\tau,\sigma}_{M\times M})^TK_{M\times M^3}+F_{M\times M^3}\}\,{}^xA^{(\sigma,a,b,c)}_{M^3\times M^3}-$$
$$\lambda_y\{(H^{\tau,\sigma}_{M\times M})^TK_{M\times M^3}+F_{M\times M^3}\}\,{}^yA^{(\sigma,a,b,c)}_{M^3\times M^3}-$$
$$\lambda_z\{(H^{\tau,\sigma}_{M\times M})^TK_{M\times M^3}+F_{M\times M^3}\}\,{}^zA^{(\sigma,a,b,c)}_{M^3\times M^3}-F^o_{M\times M^3}]=0.$$

In generalized form we can write it as

$$A_{M\times M}K_{M\times M^3}B^T_{M^3\times M^3}-K_{M\times M^3}+C_{M\times M^3}=0. \tag{4. 50}$$

Where

$$A_{M\times M}=(\frac{\lambda_x}{\chi_t}I_{M\times M}+\frac{\lambda_y}{\chi_t}I_{M\times M}+\frac{\lambda_z}{\chi_t}I_{M\times M})(H^{\tau,\sigma}_{M\times M})^T,$$

$$B^T_{M^3\times M^3}=\ {}^xA^{(2,a,b,c)}_{M^3\times M^3}+\ {}^yA^{(2,a,b,c)}_{M^3\times M^3}+\ {}^zA^{(2,a,b,c)}_{M^3\times M^3},$$

and

$$C_{M\times M^3}=F_{M\times M^3}\{\frac{\lambda_x}{\chi_t}\,{}^xA^{(2,a,b,c)}_{M^3\times M^3}+\frac{\lambda_y}{\chi_t}\,{}^yA^{(2,a,b,c)}_{M^3\times M^3}+\frac{\lambda_z}{\chi_t}\,{}^zA^{(2,a,b,c)}_{M^3\times M^3}\}+F^o_{M\times M^3}.$$

The resulting equation (4. 50) is an algebraic equation of Sylvester type and can be easily solved for the unknown matrix $K_{M\times M^3}$ by using the MatLab command *dlyap*. By using the value of K in (4. 45) we can easily obtain the approximate solution of the problem.

5. Illustrative Examples

The method mentioned above is the extension of pseudo spectral method. The basic property of such method is that the accuracy of the solutions depend on the smoothness of the solutions. If the exact solution of the problem is smooth then the method will yield more accurate solution at small scale level. However, if the solution is not smooth, it will be needed to simulate the algorithm at relatively higher scales. As experiment, we approximate the solution of two different problems. In the first problem the source term is zero. However for the second problem, we select a suitable source term such that the exact solution of the problem is known.

EXAMPLE 1. *Consider the following time fractional heat conduction problem*

$$\chi_t \frac{\partial^\sigma u(t,x,y,z)}{\partial t^\sigma} = \lambda_x \frac{\partial^2 u(t,x,y,z)}{\partial x^2} + \lambda_y \frac{\partial^2 u(t,x,y,z)}{\partial y^2} + \lambda_z \frac{\partial^2 u(t,x,y,z)}{\partial z^2} + I(t,x,y,z), \qquad u(0,x,y,z) = f(x,y,z). \tag{5. 51}$$

Choose $\chi_t = \lambda_x = \lambda_y = \lambda_z = 1$, $0 < \sigma \leq 1$, $t \in [0,1]$, $x \in [0,1]$,$y \in [0,1]$ *and* $z \in [0,1]$ *and the initial condition* $u(0,x,y,z) = e^{(x+y+z)}$. *Taking the source term* $I(t,x,y,z) = 0$, *then the exact solution of the problem for fix* $\sigma = 1$ *is* $u(t,x,y,z) = e^{(x+y+z+3t)}$. *To check the accuracy of the scheme we fix* $\sigma = 1$*(because the exact solution at* $\sigma = 1$ *is known) and simulate the algorithm at different scale level. We observe that the accuracy of the solution increases as the scale level increase. At scale level* $M = 10$, *we observe that the approximate solution is equal to the exact solution with difference less than* 10^{-3}. *We compare the exact solution with the approximate solution for different values of* t. *In Fig* (1), *we fix* $t = 0.3$ *and compare the exact solution with the approximate solution for values of* z, *that is,* $z = [0.1, 0.3, 0.7, 0.9]$. *The surfaces in the Fig* (1) *represents the approximate solution at some fix* z, *while the color dots represents the exact solution at the corresponding values of* z. *Fig* (2) *and Fig* (3) *shows the same phenomena at some other values of* t, *that is,* $t = 0.5$ *and* $t = 0.9$ *respectively. The most interesting property of fractional differential equation is that the solution at fractional values approaches to the solution at integer values as the order of derivative approaches from fractional to integer simultaneously. We use this property to show that the method provide accurate solution at fractional values. For this purpose, we approximate the solution at different value of* σ *and observe that as* $\sigma \to 1$ *the solution approaches to the exact solution. In Fig* (4) *and Fig* (5), *we show this behavior of the solution at two different points of yz-plane. One can easily note that the approximate solution approaches to the exact solution(color dots). We observe that for the current problem the method provide much more accurate results. We approximate the absolute error at two different points of the yz-plane, as shown in the Fig* (6) *and Fig* (7) *the absolute error is less than* 10^{-5} *which is relatively accepted number for such complicated problems. The simulations of this example is carried out with selecting the parameter of the Jacobi polynomials* $\xi = \zeta = 1$.

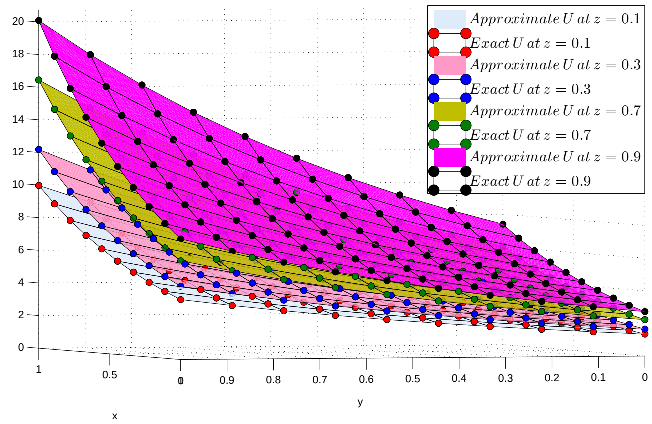

Fig. 1 : *The comparison of the exact and approximate solution of example* 1 *at* $t = 0.3$ $\xi = \zeta = 1$, $M = 10$, $a = 1$, $b = 1$, $c = 1$, $\tau = 1$ *and the order of derivative* $\sigma = 1$.

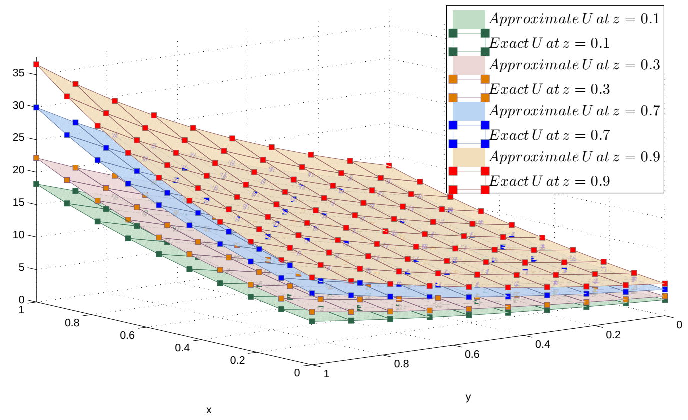

Fig. 2 The comparison of the exact and approximate solution of example 1 at $t = 0.5$, $\xi = \zeta = 1$, $M = 10$, $a = 1, b = 1, c = 1, \tau = 1$ *and the order of derivative* $\sigma = 1$.

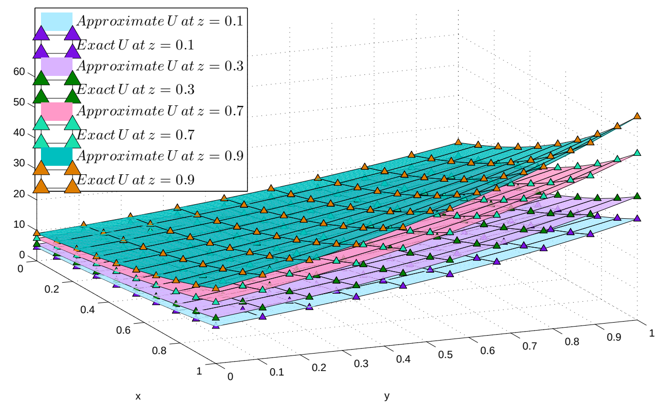

Fig. 3 The comparison of the exact and approximate solution of example 1 at $t = 0.9$ $\xi = \zeta = 1$, $M = 10$, $a = 1$, $b = 1$, $c = 1$, $\tau = 1$ *and the order of derivative* $\sigma = 1$.

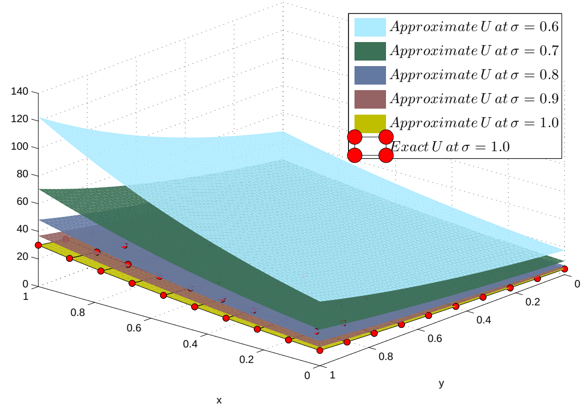

Fig. 4 The approximate solution of example 1 at fractional value of σ (surface). We fix $z = 0.3$, $y = 0.3$, $M = 10$,$a = 1, b = 1, c = 1, \tau = 1$.

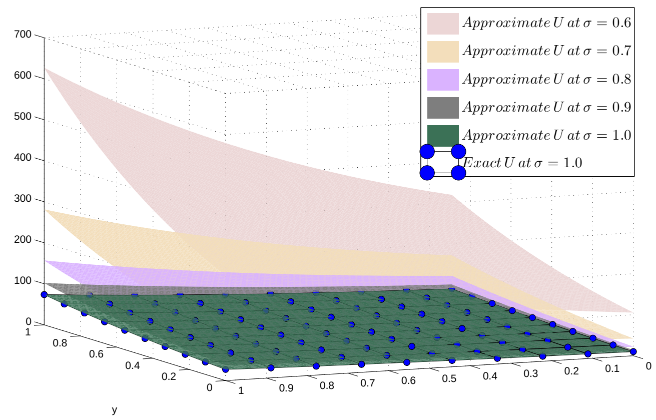

Fig. 5: The approximate solution of example 1 at fractional value of σ.The blue dots represents the exact solution at $\sigma = 1$.

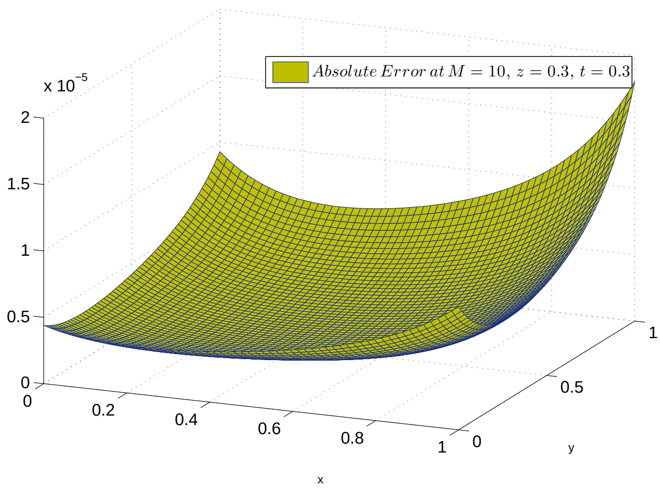

Fig. 6 : The absolute error of example 1 at $M = 10$.Here we fix $z = 0.3$, $t = 0.3$, $\sigma = 1$, $a = 1, b = 1, c = 1, \tau = 1$.

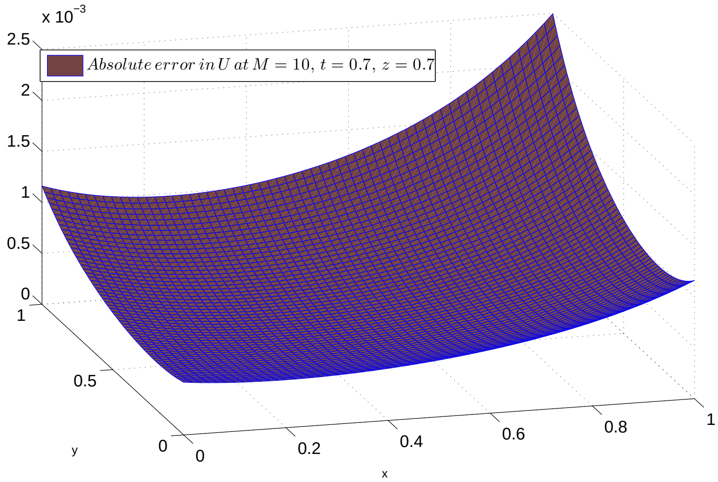

Fig. 7 : *The absolute error of example 1 at* $M = 10$.*Here we fix* $z = 0.7$, $t = 0.7$, $\sigma = 1$, $a = 1, b = 1, c = 1, \tau = 1$.

EXAMPLE 2. *Consider the time fractional heat conduction problem*

$$\chi_t \frac{\partial^\sigma u(t,x,y,z)}{\partial t^\sigma} = \lambda_x \frac{\partial^2 u(t,x,y,z)}{\partial x^2} + \lambda_y \frac{\partial^2 u(t,x,y,z)}{\partial y^2} + \lambda_z \frac{\partial^2 u(t,x,y,z)}{\partial z^2} + I(t,x,y,z),$$
$$u(0,x,y,z) = f(x,y,z). \tag{5. 52}$$

Also choose $\chi_t = \lambda_x = \lambda_y = \lambda_z = 1$, $0 < \sigma \leq 1$, $t \in [0,1]$, $x \in [0,1]$, $y \in [0,1]$ *and* $z \in [0,1]$.*If we let the initial condition*

$$u(0,x,y,z) = x^2\, y^2\, z^2 - 2\,x + y + z$$

and take the source term

$$I(t,x,y,z) = 2\,x\,y\,(t\,x\,y + x\,y\,z) - 2\,(t\,x + x\,z)^2 - 2\,(t\,y + y\,z)^2 - 2\,x^2\,y^2 - 6\,t^3\,x\,z^3 - 6\,t^3\,x^3\,z + 3\,t^2\,x^3\,z^3 - 12\,t^4\,x^2\,y^4\,z^4 - 12\,t^4\,x^4\,y^2\,z^4 - 12\,t^4\,x^4\,y^4\,z^2 + 4\,t^3\,x^4\,y^4\,z^4$$

then the exact solution of the problem for $\sigma = 1$ *is*

$$u(t,x,y,z) = y - 2\,x + z + (t\,x\,y + x\,y\,z)^2 + t^3\,x^3\,z^3 + t^4\,x^4\,y^4\,z^4.$$

We approximate the solution of this problem with the new technique and as expected we get a high accuracy of the approximate solution. We observe that at scale level $M = 6$ *(which is much more small scale for such problem) the approximate solution is equal to the exact solution with maximum difference less than* 10^{-15} *which is highly negligible. We compare the approximate solution with the exact solution at three different value of* t. *The comparison at* $t = 0.3$ *is displayed in Fig.* (8), *while at* $t = 0.5$ *and* $t = 0.9$ *is shown in Fig.* (9) *and Fig* (10).*One can easily note that the exact solution matches very well with the approximate solution.Note that the dots in these figures represents the exact solution and the surface in these figures represents the approximate solution.We also approximate the solution for fractional value of* σ, *and as expected, the approximate solution approaches the exact solution as the order of derivative* $\sigma \to 1$.*Fig* (11) *and Fig* (12) *shows this phenomena at some fixed points of the yz-plane. The absolute amount of error is displayed in Fig* (13) *and Fig* (14) *at some point of the yz-plane.One can see that the absolute amount* *of error is less than* 10^{-16}*. This example is analyzed with choosing the parameters of the Jacobi polynomials as* $\xi = \zeta = 0$*.*

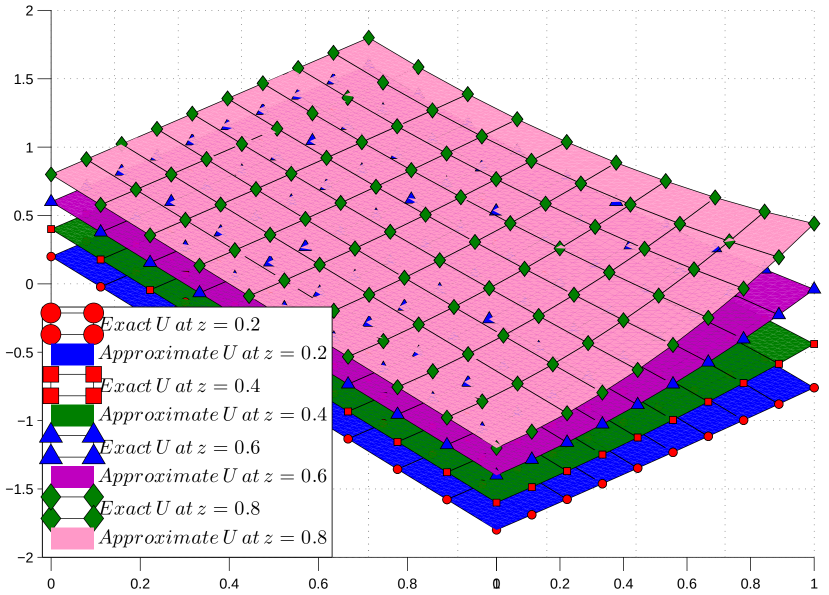

Fig. 8 The comparison of the exact and approximate solution example 2 at $t = 0.3$*,* $\xi = \zeta = 0$*,* $M = 6$*,* $a = 1, b = 1, c = 1, \tau = 1$ *and the order of derivative* $\sigma = 1$*.*

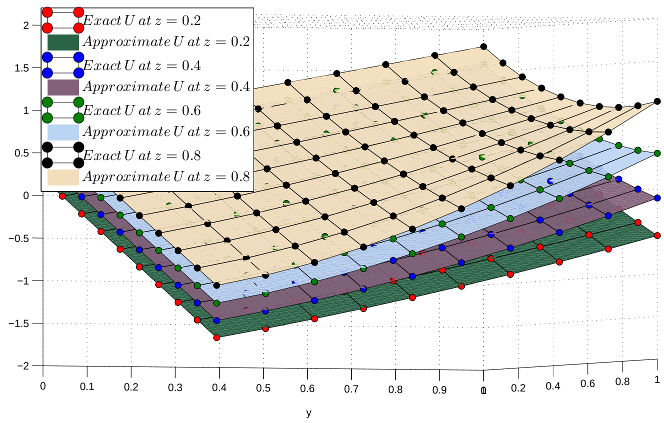

Fig. 9 The comparison of the exact and approximate solution example 2 at $t = 0.5$*,* $\xi = \zeta = 0$*,* $M = 6$*,* $a = 1, b = 1, c = 1, \tau = 1$ *and the order of derivative* $\sigma = 1$*.*

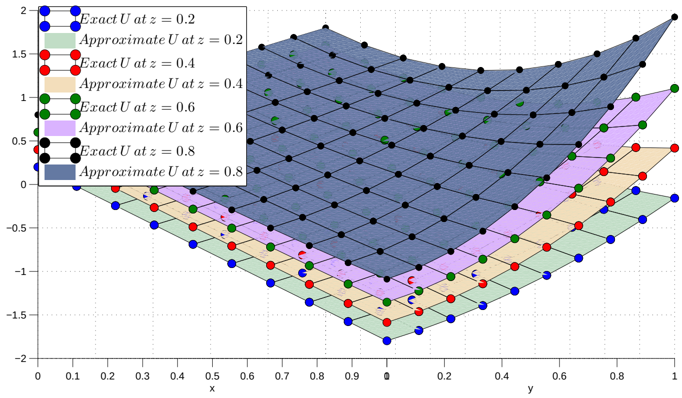

Fig. 10 *The comparison of the exact and approximate solution example 2 at* $t = 0.9$, $\xi = \zeta = 0$, $M = 6$, $a = 1, b = 1, c = 1, \tau = 1$ *and the order of derivative* $\sigma = 1$.

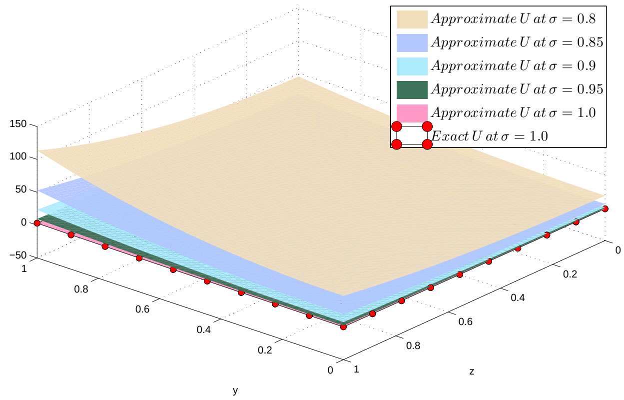

Fig. 11 *The approximate solution of example 2 at fractional value of* σ *(surface). Here we fix* $x = 0.3$, $t = 0.3$, $M = 6$,$a = 1, b = 1, c = 1, \tau = 1$.

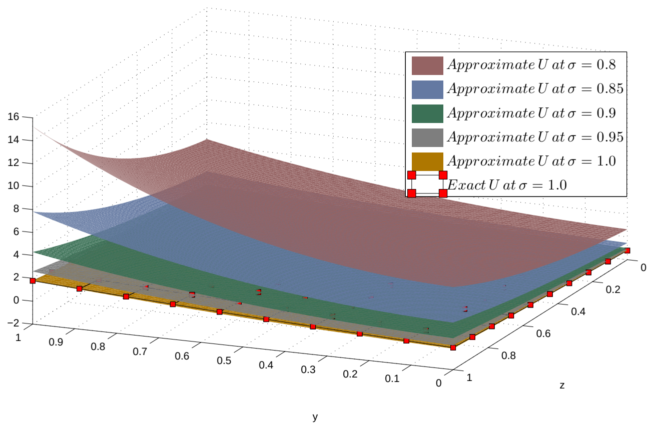

Fig. 12 *The approximate solution of example 2 at fractional value of σ (surface). Here we fix $x = 0.7$, $t = 0.7$, $M = 6$,$a = 1, b = 1, c = 1, \tau = 1$.*

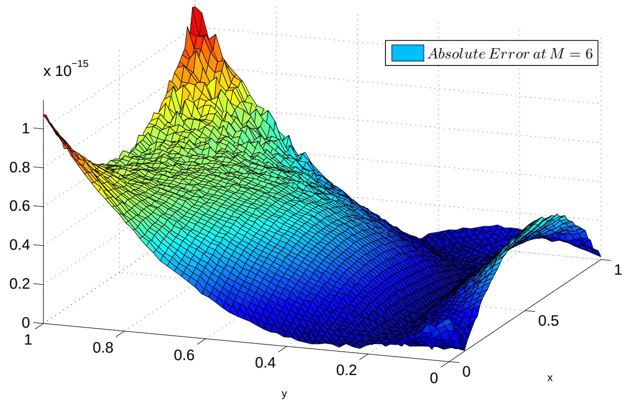

Fig. 13 *The absolute error of example 2 at $M = 6$, we fix $z = 0.4$, $t = 0.4$.*

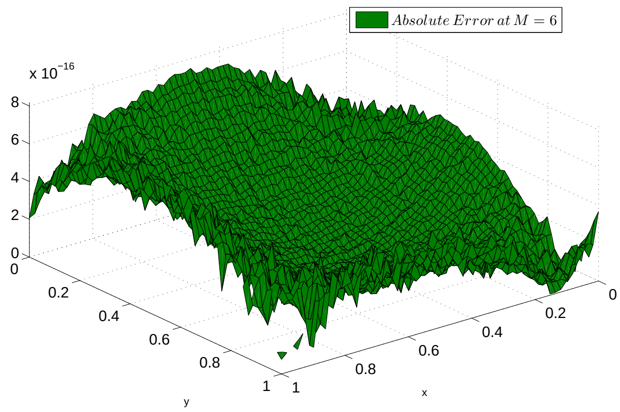

Fig. 14 *The absolute error of example 2 at $M = 6$, we fix $z = 0.8$, $t = 0.8$*

EXAMPLE 3. *Consider the following integer order heat conduction problem*

$$\frac{\partial U(t,x,y,z)}{\partial t} = \frac{\partial^2 U(t,x,y,z)}{\partial x^2} + \frac{\partial^2 U(t,x,y,z)}{\partial y^2} + \frac{\partial^2 U(t,x,y,z)}{\partial z^2} \quad (5.\ 53)$$
$$u(0,x,y,z) = f(x,y,z).$$

where $t \in [0,1]$, $x \in [0,1]$,$y \in [0,1]$ and $z \in [0,1]$.If we let the initial condition $U(0,x,y,z) = (1-y)e^{(x+z)}$, then the exact solution of the problem is

$$U(t,x,y,z) = (1-y)e^{(x+z+2t)}.$$

This problem is also solved in [2] *using homotopy analysis method and variational iteration method. We approximate the solution of this problem with our new technique.And as expected we found that the approximate solution matches very well with the exact solution. The comparison of exact and approximate solution at some fixed value of t,ie $t = 0.3, 0.6, 0.9$ and at each value of t the solution is displayed at fix value of z.$Fig(15)$,$Fig(16)$ and $Fig(17)$ shows the comparison of exact and approximate solution at $t = 0.3, 0.6\ and\ 0.9$*

respectively. Note that here we fix the scale level $M = 9$. We observe that the method yields a very high accurate estimate of the solution. And the error of approximation (absolute error) decreases significantly by the increase of the scale level M. In order to compare the accuracy of the method we compare our results with the analytic solution obtained in [2]. *We calculate the absolute difference of the exact and approximate solution using the current method. We also calculate the difference of exact solution and n-th order analytic solution reported in* [2]. *We observe that the error obtained with this new method in much more less than that reported in* [2]. *$Fig(18)$ and $Fig(19)$ shows comparison of absolute error at two different points of the space.*

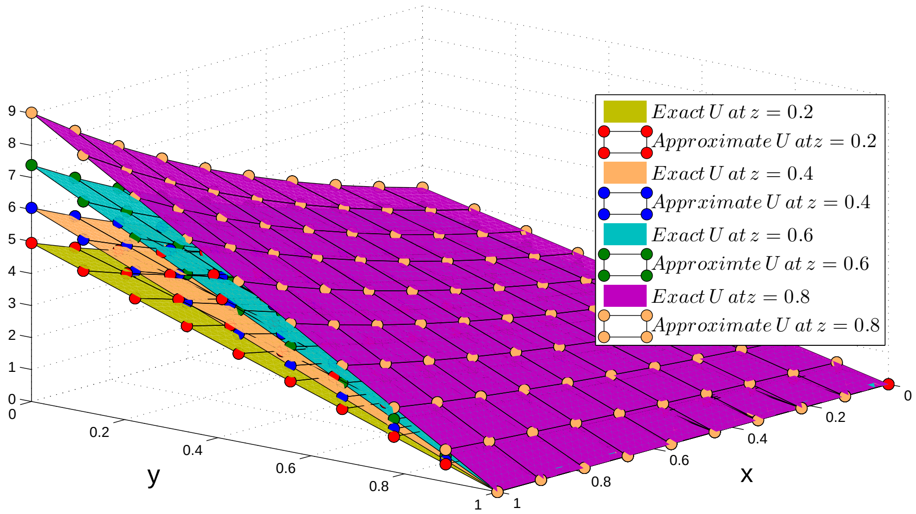

Fig. 15 *The approximate solution of example 3 at different value of z where $\sigma = 1$, $M = 9$,$t = 0.3$.*

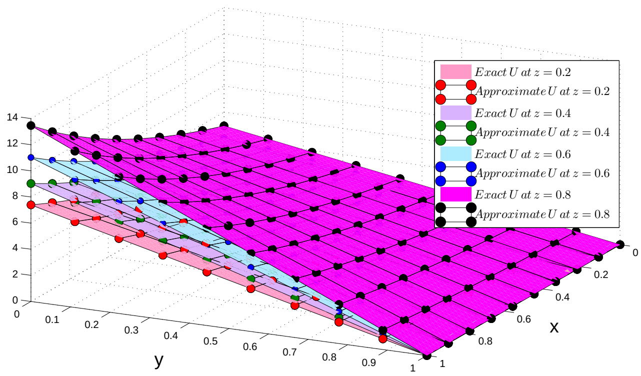

Fig. 16 *The approximate solution of example 3 at different value of z where $\sigma = 1$, $M = 9$,$t = 0.6$.*

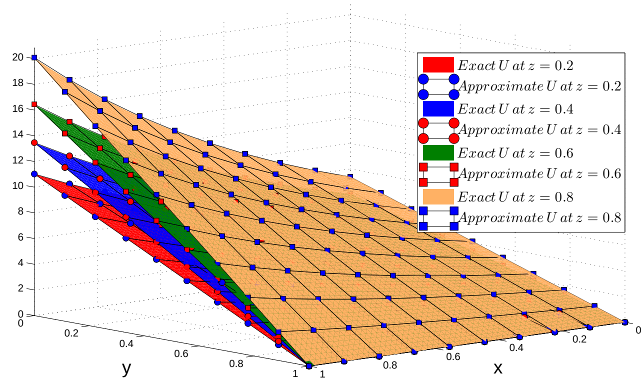

Fig. 17 *The approximate solution of example 3 at different value of z where $\sigma = 1$,* $M = 9$,$t = 0.9$.

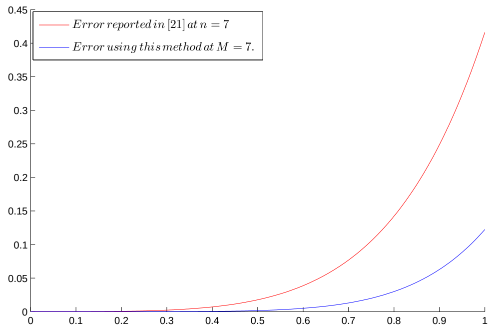

Fig. 18 *Comparison of absolute error of example at* $(x, y, z) = (0.5, 0.5, 0.5)$ *at* $M = 7$.

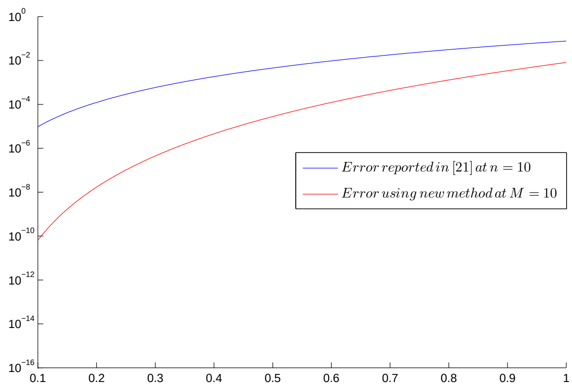

Fig. 19 *Comparison of absolute error at* $(x, y, z) = (0.9, 0.9, 0.9)$ *at* $M = 10$.

6. CONCLUSION

From the above analysis and calculation we concluded that the method provide a very good approximation to the problems under consideration.This method can efficiently solve partial differential equation in four variables.The main advantage of the method is its high accuracy.The method can be easily extended to solve more complicated problems.We believe that one may obtain a more accurate solution by using some other kinds of orthogonal polynomials like Bernstein or Laguerre.Our future work is related to the extension of the method to solve such problems under different kinds of boundary conditions.We expect that the reader may find the work interesting and useful.

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