N-Transform – Properties and Applications

Zafar H. Khan*, Waqar. A. Khan**
*Department of Mathematics, COMSATS Institute of Information Technology, Abbottabad, Pakistan.
**Department of Engineering Sciences, PN Engineering College
National University of Sciences and Technology, Islamabad, Pakistan.

Abstract

A new integral transform similar to Laplace and Sumudu transforms is introduced. It converges to both transforms just by changing variables. A table is presented for existing Laplace and Sumudu transforms. To explain the use of N-Transform in linear differential equation, an example of unsteady fluid flow over a plane wall is presented.

Introduction

There are numerous integral transforms [1-4] to solve differential equations and integral equations. Of these, the Laplace transformation is the most widely used. In view of many interesting properties which make visualization easier, we introduced a new integral transform, termed as N-Transformation and applied it to the solution of fluid flow problem (Linear differential equations). Subsequently, we derived the N-Transform of different functions and derivatives used in engineering problems. In this paper we discussed the basic theory of N-Transform with supporting examples and presented a table. The specialty of this new transform is the convergence to the Laplace and Sumudu transform. It works as a check on these two mentioned transforms. Here we also give application of N-Transform in an unsteady flow problem of a viscous fluid due to an oscillating plane wall [5].

Definition and basic theory

Definition: Let $f(t)$ be a function defined for all $t \geq 0$.The $N-transform$ of $f(t)$ is the function $R(u,s)$ defined by

$$R(u,s) = N(f) = \int_0^\infty f(ut)e^{-st}dt \tag{1}$$

provided the integral on the right exists. The original function $f(t)$ in (1) is called the inverse transform or inverse of $R(u,s)$ and is denoted by

$$f(t)=N^{-1}\{R(u,s)\}$$

N-Transform of elementary functions

Exponential function

Let $f(t)=e^{at}$, when $t \geq 0$, where a is a constant, the N-transform of this function can be written as

$$N(e^{at}) = \int_0^\infty e^{aut}e^{-st}dt$$

$$= \lim_{b\to\infty}\int_0^b e^{-(s-au)t}dt$$

$$= \lim_{b\to\infty}\left[\frac{e^{-(s-au)t}}{s-au}\right]_0^b$$

$$= \frac{1}{s-au} \tag{2}$$

$$= \begin{cases} \dfrac{1}{s-a} & \text{Laplace transform, } (u=1) \\ \dfrac{1}{1-au} & \text{Sumudu transform, } (s=1) \end{cases}$$

Unit step function:

$$u(t)=\begin{cases} 1, & t>0 \\ 0, & t\leq 0 \end{cases}$$

$$N[u(t)] = \int_0^\infty e^{-st}dt = \lim_{b\to\infty}\left[-\frac{e^{-st}}{s}\right]_0^b = \frac{1}{s}$$

$$= \begin{cases} \dfrac{1}{s} & \text{Laplace transform, } (u=1) \\ 1 & \text{Sumudu transform, } (s=1) \end{cases}$$

Sufficient condition for the existence of N-Transform

Theorem 2.1: *If $f(t)$ is sectionally continuous in every finite interval $0 \leq t \leq K$ and of exponential order γ for $t > K$, then its N-Transform $R(u,s)$ exists for all $s > \gamma, u > \gamma$.*

Corresponding Author: Waqar. A. Khan
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Proof:
We have for any positive number K,

$$\int_0^\infty f(ut)e^{-st}dt = \int_0^K e^{-st} f(ut)dt + \int_K^\infty e^{-st} f(ut)dt$$

Since f(t) is sectionally continuous in every finite interval $0 \le t \le K$, the first integral on the right side exists. Also the second integral on the right exists, since f(t) is of exponential order γ for $t > K$. To see this we have only to observe that in such case

$$\left|\int_K^\infty e^{-st} f(ut)dt\right| \le \int_K^\infty \left|e^{-st} f(ut)\right|dt$$

$$\le \int_0^\infty e^{-st}\left|f(ut)\right|dt$$

$$\le \int_0^\infty e^{-st} M e^{\gamma ut} dt = \frac{M}{s-\gamma u}$$

Properties of N-transform

Linearity property

Theorem 2.2: If a & b are any constants and $f(t)$ and $g(t)$ are functions, then

$$N\{af(t)+bg(t)\} = aN\{f(t)\} + bN\{g(t)\}$$

Proof:

$N(f) = \int_0^\infty f(ut)e^{-st}dt$ and $N(g) = \int_0^\infty g(ut)e^{-st}dt$.
Then if a & b are any constants.

$$N\{af(t)+bg(t)\} = \int_0^\infty e^{-st}\{af(ut)+bg(ut)\}dt$$

$$= a\int_0^\infty e^{-st} f(ut)dt + b\int_0^\infty e^{-st} g(ut)dt$$

$$= aN(f) + bN(g).$$

Example:

$$N\{3t + 4\cos 2t\} = 3N(t) + 4N(\cos 2t)$$

$$= 3\frac{u}{s^2} + 4\frac{s}{s^2+4u^2}$$

$$= \begin{cases} \dfrac{3}{s^2} + \dfrac{4s}{s^2+4} & \text{Laplace transform, } (u=1) \\[2ex] 3u + \dfrac{4}{1+4u^2} & \text{Sumudu transform, } (s=1) \end{cases}$$

First translation or shifting property

Theorem 2.3: *let $f(t)$ be a continuous function & $t \ge 0$ then*

$$N\left[e^{at} f(t)\right] = \frac{s}{s-au}\left[\frac{us}{s-au}\right]$$

Proof: The *N-transform* of $e^{at} f(t)$ is given by

$$N\left[e^{at} f(t)\right] = \int_0^\infty e^{-(s-au)t} f(ut)dt$$

Therefore by change of variable $\left(w = \dfrac{s-au}{s}t\right)$ we get

$$N\left[e^{at} f(t)\right] = \frac{s}{s-au}\int_0^\infty e^{-sw} f\left(\frac{usw}{s-au}\right)dw = \frac{s}{s-au} R\left(\frac{us}{s-au}\right).$$

$$= \begin{cases} \dfrac{1}{1-au} R\left(\dfrac{u}{1-au}\right) & \text{Sumudu Transform } (s=1) \\[2ex] R(s-a) & \text{Laplace Transform } (u=1) \end{cases}$$

Change of scale property

Theorem 2.4: *If $N\{f(t)\} = R(s.u)$ then*

$$N\{f(at)\} = \frac{1}{a}R(s.u)$$

Proof:

$$N(f(at)) = \int_0^\infty e^{-st} f(aut)dt$$

$$= \int_a^\infty e^{-st} f(up)\frac{dp}{a}$$

$$= \frac{1}{a}\int_a^\infty e^{-st} f(up)dp$$

$$= \frac{1}{a}R\left(\frac{s}{a}, u\right)$$

$$= \begin{cases} R(au) & \text{Sumudu transform } (s=1) \\[2ex] \dfrac{1}{a}R\left(\dfrac{s}{a}\right) & \text{Laplace transform } (u=1) \end{cases}$$

Using the transformation $p = at$.

N-Transform of derivatives

Theorem 2.5: *If* $N\left\{f(t)\right\}=R(s.u)$ *then*

$$N\left\{f'(t)\right\}=\frac{s}{u}R(s.u)-\frac{f(0)}{u}$$

Proof:

$$N\left\{f'(t)\right\}=\int_0^\infty e^{-st}f'(ut)dt$$

$$=\lim_{p\to\infty}\int_0^p e^{-st}f'(ut)dt$$

$$=\lim_{p\to\infty}\left\{\left|\frac{e^{-st}f(ut)}{u}\right|_0^p+\frac{s}{u}\int_0^p e^{-st}f'(ut)dt\right\}$$

$$=\lim_{p\to\infty}\left\{\begin{array}{l}\frac{e^{-sp}f(up)}{u}-\frac{f(0)}{u}+\\ \frac{s}{u}\int_0^p e^{-st}f'(ut)dt\end{array}\right\}$$

$$=\frac{s}{u}\int_0^p e^{-st}f(ut)dt-\frac{f(0)}{u}$$

$$=\frac{s}{u}R(s,u)-\frac{f(0)}{u}$$

Theorem 2.6: *If* $N\left\{f(t)\right\}=R(s.u)$ *then*

$$N\left\{f''(at)\right\}=\frac{s^2}{u^2}R(s.u)-\frac{s}{u^2}f(0)-\frac{f'(0)}{u}.$$

Proof:

By theorem 2.4 $N\left\{G'(t)\right\}=\frac{s}{u}N\left\{G(t)\right\}-\frac{f(0)}{u}$

Let $G(t)=f'(t)$ Then

$$N\left\{f''(t)\right\}=\frac{s}{u}N\left\{f'(t)\right\}-\frac{f'(0)}{u}$$

$$=\frac{s}{u}\left\{\frac{s}{u}N\left\{f(t)\right\}-\frac{f(0)}{u}\right\}-\frac{f'(0)}{u}$$

$$=\frac{s^2}{u^2}N\left\{f(t)\right\}-\frac{s}{u^2}f(0)-\frac{f'(0)}{u}.$$

$$=\frac{s^2}{u^2}R(s,u)-\frac{s}{u^2}f(0)-\frac{f'(0)}{u}.$$

The generalization to higher order derivatives can be proved by using mathematical induction.

N-Transform of integrals

Theorem 2.7: *If* $N\left\{f(t)\right\}=R(s.u)$ *then*

$$N\left\{\int_0^t f(p)dp\right\}=\frac{u}{s}R(s,u)$$

Proof:

Let $G(t)=\int_0^t f(p)dp$. Then $G'(t)=f(t)$ and $G(0)=0$. Taking the *N-Transform* of both sides, we have

$$N\left\{G'(t)\right\}=\frac{s}{u}N\left\{G(t)\right\}-\frac{f(0)}{u}$$

$$=\frac{s}{u}N\left\{G(t)\right\}=R(s,u)$$

Thus

$$\frac{s}{u}N\left\{G(t)\right\}=R(s,u)\Rightarrow N\left\{G(t)\right\}=\frac{u}{s}R(s,u)$$

$$N\left\{\int_0^t f(p)dp\right\}=\frac{u}{s}R(s,u)$$

Application of N-Transform in fluid flow problem [5]

The governing equation for the flow over a plane wall which is initially at rest, it sets in motion due to the oscillation of the plane wall and the fluid stays in the region of $y\geq 0$ and the $x-axis$ is chosen as the plane wall [5] is

$$\frac{\partial u}{\partial t}=\upsilon\frac{\partial^2 u}{\partial y^2}\qquad(3)$$

With an initial and boundary conditions are

$$u(0,y)=0,u(t,\infty)=0, \textit{and } u(t,0)=U\cos\omega t\qquad(4)$$

Now we will take *N-Transformation* with respect to "t" with the notation that $N[u(t,y)]=R(s,u,y)$ of equation (3) and of the boundary conditions of equation (4)

(1)implies that $N\left[\frac{\partial u}{\partial t}\right]=N\left[\upsilon\frac{\partial^2 u}{\partial y^2}\right]$ (5)

$$N\left[\frac{\partial u}{\partial t}\right]=\upsilon N\left[\frac{\partial^2 u}{\partial y^2}\right].$$

$$N\left[\frac{\partial u}{\partial t}\right]=\frac{s}{u}R(s,u,y)-\frac{u(0,y)}{u}$$

$$= \frac{s}{u} R(s,u,y) - 0$$

$$= \frac{s}{u} R(s,u,y) \qquad (6)$$

$$N\left[\frac{\partial^2 u}{\partial y^2}\right] = \frac{\partial^2 [R(s,u,y)]}{\partial y^2} \qquad (7)$$

$$= R''(s,u,y)$$

Now we will take *N-Transformation* of boundary conditions.

$$N[u(\infty, y)] = N[0]$$
$$R(s,u,\infty) = 0.$$

$$N[u(t,0)] = N[U \cos \omega t]$$
$$R(s,u,0) = UN[\cos \omega t]$$
$$R(s,u,0) = U \frac{s}{s^2 + \omega^2 u^2}$$

By putting equation (6) and equation (7) in equation (5)

$$\frac{s}{u} R(s,u,y) = \upsilon R''(s,u,y)$$
$$\frac{d^2}{dy^2} R(s,u,y) - \frac{s}{\upsilon u} R(s,u,y) = 0$$

The characteristic equation of the above differential equation is

$$D^2 - \frac{s}{\upsilon u} = 0 \Rightarrow D = \pm\sqrt{\frac{s}{\upsilon u}}$$

Since the roots are real and distinct therefore the solution is of the form

$$R(s,u,y) = c_1 e^{\sqrt{\frac{s}{u\upsilon}}y} + c_2 e^{-\sqrt{\frac{s}{u\upsilon}}y} \qquad (8)$$

Now we will find $c_1 and c_2$ by using boundary conditions Eq. (8).

First boundary gives

$$R(s,u,\infty) = c_1 e^{\infty} + c_2 e^{\infty}$$

To make it bounded let $c_1 = 0$ thus

$$R(s,u,y) = c_2 e^{-\sqrt{\frac{s}{u\upsilon}}y} \qquad (9)$$

Second boundary condition gives

$$R(s,u,0) = c_2 e^0 = U \frac{s}{s^2 + \omega^2 u^2} \Rightarrow c_2 = U \frac{s}{s^2 + \omega^2 u^2}$$

Now we will put the valves of $c_1 and c_2$ in equation (8) we get

$$R(s,u,y) = U \frac{s}{s^2 + \omega^2 u^2} e^{-\sqrt{\frac{s}{u\upsilon}}y}$$

Now using partial fraction we will get

$$R(s,u,y) = U \frac{1}{2}\left[\frac{\exp(-\sqrt{\frac{s}{u\upsilon}}y)}{s + i\omega u} + \frac{\exp(-\sqrt{\frac{s}{u\upsilon}}y)}{s - i\omega u}\right]$$

$$= \begin{cases} U\frac{1}{2}\left[\frac{\exp\left(-\sqrt{\frac{s}{\upsilon}}y\right)}{s+i\omega} + \frac{\exp\left(-\sqrt{\frac{s}{\upsilon}}y\right)}{s-i\omega}\right] & \text{Laplace transform } (u=1) \\ U\frac{1}{2}\left[\frac{\exp\left(-\sqrt{\frac{1}{\upsilon u}}y\right)}{1+i\omega u} + \frac{\exp\left(-\sqrt{\frac{1}{\upsilon u}}y\right)}{1-i\omega u}\right] & \text{Sumudu transform } (s=1) \end{cases}$$

Whose inverse Laplace Transform is

$$\frac{u}{U} = \frac{1}{4}\left[\exp(i\omega t)\right] \times \upsilon \left[\exp\left(-y\sqrt{\frac{i\omega}{\upsilon}}\right) erfc\left(\frac{y}{2\sqrt{\upsilon t}} - \sqrt{i\omega t}\right)\right] +$$
$$\left[\exp\left(y\sqrt{\frac{i\omega}{\upsilon}}\right) erfc\left(\frac{y}{2\sqrt{\upsilon t}} + \sqrt{i\omega t}\right)\right]$$
$$+\frac{1}{4}\left[\exp(-i\omega t)\right]\left[\exp\left(-y\sqrt{-\frac{i\omega}{\upsilon}}\right) erfc\left(\frac{y}{2\sqrt{\upsilon t}} - \sqrt{-i\omega t}\right)\right] +$$
$$\left[\exp\left(-y\sqrt{-\frac{i\omega}{\upsilon}}\right) erfc\left(\frac{y}{2\sqrt{\upsilon t}} + \sqrt{-i\omega t}\right)\right] \qquad (9)$$

Using non-dimensional variables and for large time, [5] obtained the following steady state solution.

$$\frac{u}{U} = \left[\exp\left(-\frac{\eta}{\sqrt{2}}\right)\right]\left[\cos\left(\tau - \frac{\eta}{\sqrt{2}}\right)\right]$$

The dimensionless velocity for the steady state condition is shown in Fig. 1. The velocity profiles are obtained for τ=0,1 , 2, and 3 seconds. These profiles show the oscillatory motion of the plate.

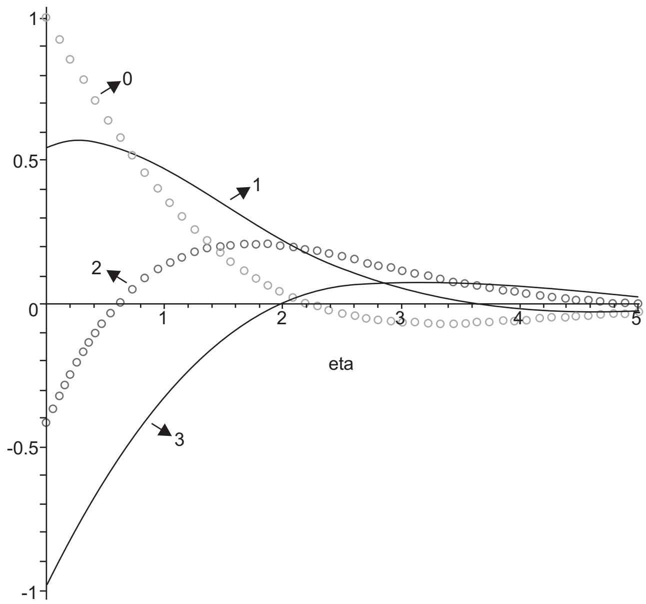

Fig. 1. Dimensionless velocity distribution for various values of time.

Conclusion

A new transform is introduced and its properties are mentioned. It is demonstrated that the theorems related to Laplace and Sumudu are also true for this new transform, Finally, the new transformed is applied to a fluid flow problem and the results are compared with the existing solution.

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Appendix

Special N-transforms and the conversation to Sumudu and Laplace

S.No.	$f(t)$	$N[f(t)]$	$S[f(t)]$	$L[f(t)]$
1	1	$\frac{1}{s}$	1	$\frac{1}{s}$
2	t	$\frac{u}{s^2}$	u	$\frac{1}{s^2}$
3	e^{at}	$\frac{1}{s-au}$	$\frac{1}{1-au}$	$\frac{1}{s-a}$
4	$\frac{1}{\omega}\sin \omega t$	$\frac{u}{s^2+\omega^2 u^2}$	$\frac{u}{1+\omega^2 u^2}$	$\frac{1}{s^2+\omega^2}$
5	$\cos \omega t$	$\frac{s}{s^2+\omega^2 u^2}$	$\frac{1}{1+\omega^2 u^2}$	$\frac{s}{s^2+\omega^2}$
6	$\cosh t$	$\frac{s}{s^2-u^2}$	$\frac{1}{1-u^2}$	$\frac{s}{s^2-1}$
7	$\frac{t^{n-1}}{(n-1)!}, n=1,2$	$u^{(n-1)} s^{(-1n)}$	$u^{(n-1)}$	$s^{(-1n)}$

S.No.	$f(t)$	$N[f(t)]$	$S[f(t)]$	$L[f(t)]$
8	$\frac{t^{n-1}}{\Gamma(n)}, n>0$	$u^{(n-1)}s^{(-1n)}$	$u^{(n-1)}$	$s^{(-1n)}$
9	$\cos t$	$\frac{s}{s^2+u^2}$	$\frac{1}{1+u^2}$	$\frac{s}{s^2+1}$
10	$\sin t$	$\frac{u}{s^2+u^2}$	$\frac{u}{1+u^2}$	$\frac{1}{s^2+1}$
11	$\sinh t$	$\frac{au}{s^2-a^2u^2}$	$\frac{au}{1-a^2u}$	$\frac{a}{s^2-a^2}$
12	$\cosh at$	$\frac{s}{s^2-a^2u^2}$	$\frac{1}{1-a^2u^2}$	$\frac{s}{s^2-a^2}$
13	$e^{bt}\cosh at$	$\frac{s-bu}{(s-bu)^2-a^2u^2}$	$\frac{1-bu}{(1-bu)^2-a^2u^2}$	$\frac{s-b}{(s-b)^2-a^2}$
14	$\frac{e^{bt}\sinh at}{a}$	$\frac{u}{(s-bu)^2-a^2u^2}$	$\frac{u}{(1-bu)^2-a^2u^2}$	$\frac{1}{(s-b)^2-a^2}$
15	$\frac{t\sin at}{2a}$	$\frac{su^2}{(s^2+a^2u^2)^2}$	$\frac{u^2}{(1+a^2u^2)^2}$	$\frac{s}{(s^2+a^2)^2}$
16	$t\cos at$	$\frac{u(s^2-a^2u^2)}{(s^2+a^2u^2)^2}$	$\frac{u(1-a^2u^2)}{(1+a^2u^2)^2}$	$\frac{(s^2-a^2)}{(s^2+a^2)^2}$
17	$\frac{\sin at+at\cos at}{2a}$	$\frac{s^2u}{(s^2+a^2u^2)^2}$	$\frac{u}{(1+a^2u^2)^2}$	$\frac{s^2}{(s^2+a^2)^2}$
18	$\cos at-\frac{1}{2}at\sin at$	$\frac{s^3}{(s^2+a^2u^2)^2}$	$\frac{1}{(1+a^2u^2)^2}$	$\frac{s^3}{(s^2+a^2)^2}$
19	$\frac{\sin at-at\cos at}{2a^3}$	$\frac{u^3}{(s^2+a^2u^2)^2}$	$\frac{u^3}{(1+a^2u^2)^2}$	$\frac{1}{(s^2+a^2)^2}$
20	$\frac{at\cos at-\sinh t}{2a^3}$	$\frac{u^3}{(s^2-a^2u^2)^2}$	$\frac{u^3}{(1-a^2u^2)^2}$	$\frac{1}{(s^2-a^2)^2}$
21	$\frac{t\sinh at}{2a}$	$\frac{su^2}{(s^2-a^2u^2)^2}$	$\frac{u^2}{(1-a^2u^2)^2}$	$\frac{s}{(s^2-a^2)^2}$
22	$\frac{\sinh at+at\cosh at}{2a}$	$\frac{s^2u}{(s^2-a^2u^2)^2}$	$\frac{u}{(1-a^2u^2)^2}$	$\frac{s^2}{(s^2-a^2)^2}$

S.No.	$f(t)$	$N[f(t)]$	$S[f(t)]$	$L[f(t)]$
23	$\cosh at + \frac{1}{2} at \sinh at$	$\frac{s^3}{(s^2 - a^2u^2)^2}$	$\frac{1}{(1 - a^2u^2)^2}$	$\frac{s^3}{(s^2 - a^2)^2}$
24	$t \cosh at$	$\frac{u(s^2 + a^2u^2)}{(s^2 - a^2u^2)^2}$	$\frac{u(1 + a^2u^2)}{(1 - a^2u^2)^2}$	$\frac{(s^2 + a^2)}{(s^2 - a^2)^2}$
25	$\frac{(3 - a^2t^2)\sin at - 3at \cos at}{8a^5}$	$\frac{u^5}{(s^2 + a^2u^2)^3}$	$\frac{u^5}{(1 + a^2u^2)^3}$	$\frac{1}{(s^2 + a^2)^3}$
26	$\frac{(3 - a^2t^2)\sin at + 5at \cos at}{8a}$	$\frac{s^4 u}{(s^2 + a^2u^2)^3}$	$\frac{u}{(1 + a^2u^2)^3}$	$\frac{s^4}{(s^2 + a^2)^3}$
27	$\frac{(8 - a^2t^2)\cos at - 7at \sin at}{8}$	$\frac{s^5}{(s^2 + a^2u^2)^3}$	$\frac{1}{(1 + a^2u^2)^3}$	$\frac{s^5}{(s^2 + a^2)^3}$
28	$\frac{t^2 \sin at}{2a}$	$\frac{u^3(3s^2 - a^2)}{(s^2 + a^2u^2)^3}$	$\frac{u^3(3 - a^2)}{(1 + a^2u^2)^3}$	$\frac{(3s^2 - a^2)}{(s^2 + a^2)^3}$
29	$\frac{1}{2} t^2 \cos at$	$\frac{u^2(s^3 - 3a^2u^2s)}{(s^2 + a^2u^2)^3}$	$\frac{u^2(1 - 3a^2u^2)}{(1 + a^2u^2)^3}$	$\frac{(s^3 - 3a^2s)}{(s^2 + a^2)^3}$
30	$\frac{1}{24a} t^3 \sin at$	$\frac{u^4(s^3 - a^2u^2s)}{(s^2 + a^2u^2)^4}$	$\frac{u^4(1 - a^2u^2)}{(1 + a^2u^2)^4}$	$\frac{(s^3 - a^2s)}{(s^2 + a^2)^4}$