Application of Adomian Decomposition Method for Sumudu Transform

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Abstract

In this paper, Adomian decomposition method (ADM) is employed to compute Sumudu transform of some typical functions. In this method, the solution is found in the form of a convergent power series with easily computed components. To show the efficiency of the method, numerical examples are presented and a MAPLE code is provided that computes Sumudu transform of typical functions quickly. It is demonstrated that like HPM, this method is also quite easy and fast to compute Sumudu transforms.

Introduction

The decomposition method presented by Adomian [1, 2] has been proved by many authors [3-10] to be reliable and promising. It can be used for all types of differential equations, linear or nonlinear, homogeneous or non-homogeneous. This method has many advantages over other classical methods, it avoids perturbation in order to find solutions of given nonlinear equations and provides an immediate and convergent solution without any need for linearization or discretization.

In this paper, we employed ADM to compute Sumudu transform of some analytical functions. Recently, Abbasbandy [11] and Babolian et al. [12] have computed Laplace transforms of some simple functions by homotopy perturbation method (HPM) and Adomian decomposition method (ADM), whereas, Khan et al. [13] employed HPM to obtain Sumudu transforms of some typical functions which were difficult to obtain through regular definition of Sumudu transform. They also provided a code in MAPLE 10, which is the most powerful mathematical computation engine, to compute Sumudu transform of some typical functions. A brief description of this method is given below:

Consider the first order differential equation

$$u'(x) + R(x)u = Q(wx) \tag{1}$$

with boundary condition

$$u(0) = 0 \tag{2}$$

The solution of Eq. (1) with boundary condition (2) is given by

$$u(x)f(x) = \int Q(wx)f(x)dx \tag{3}$$

where

$$f(x) = e^{\int R(x)dx}$$

The solution, given by Eq. (3), can be obtained by ADM by using $L = \frac{d}{dx}$ in Eq. (1), i.e.

$$Lu(x) + Ru(x) = Q(wx) \tag{4}$$

Eq. (4) can be rewritten in the following form

$$u(x) = \frac{Q(wx)}{R} - \frac{L}{R}u(x) \tag{5}$$

Using ADM, the solution of Eq. (5) can be written in series form as

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{6}$$

where the terms $u_n(x)$ can be determined as follows

$$\begin{aligned}
u_0 &= \frac{Q(wx)}{R} \\
u_1 &= -\frac{1}{R}L\left(\frac{Q(wx)}{R}\right) \\
u_2 &= (-1)^2\frac{1}{R^2}L^2\left(\frac{Q(wx)}{R}\right) \\
u_3 &= (-1)^3\frac{1}{R^3}L^3\left(\frac{Q(wx)}{R}\right) \\
\vdots &\qquad \vdots \\
u_n &= (-1)^n\frac{1}{R^n}L^n\left(\frac{Q(wx)}{R}\right) \qquad n = 1, 2, \ldots
\end{aligned}$$

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Sumudu transform

If Eq. (3) is considered as a definite integral from zero to infinity, then right hand side of that equation defines the Sumudu transform [14] of $Q(wx)$, i.e.

$$G(w) = S[Q(x)] = \int_0^\infty e^{-x} Q(wx)dx$$

$$= \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right] \Bigg|_{x=0}^{x=\infty} \quad (7)$$

If we take $R(x) = -1$ in Eq. (3), then $f(x)$ satisfies Eq. (7) and can be written as

$$f(x) = e^{\int R(x)dx} = e^{-x}$$

and from Eq. (3) and Eq. (6) we get

$$\int Q(wx)e^{-x}dx = \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right]$$

where $u_n(x)$ are

$$u_0 = -Q(wx)$$
$$u_1 = -L(Q(wx))$$
$$u_2 = -L^2(Q(wx))$$
$$u_3 = -L^3(Q(wx))$$
$$\vdots \quad \vdots$$
$$u_n = -L^n(Q(wx)) \qquad n = 1, 2, ...$$

For clarification, some examples are presented below:

Example 1

If $Q(x) = 1$, then according to definition of Sumudu transform $Q(wx) = 1$, and

$$S[Q(x)] = \int_0^\infty e^{-x} Q(wx)dx$$

$$= \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right] \Bigg|_{x=0}^{x=\infty} \quad (8)$$

where

$$u_0 = -Q(wx) = -1$$
$$u_1 = -L(Q(wx)) = 0$$
$$u_2 = -L^2(Q(wx)) = 0$$
$$u_3 = -L^3(Q(wx)) = 0$$
$$\vdots \quad \vdots$$
$$u_n = -L^n(Q(wx)) = 0 \qquad n = 1, 2, ...$$

By definition of Sumudu transform

$$S[1] = \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right] \Bigg|_{x=0}^{x=\infty}$$
$$= 1$$

Example 2

Let $Q(x) = x^n$, then $Q(wx) = (wx)^n$, and

$$S[x^n] = \int_0^\infty e^{-x} (wx)^n dx$$

$$= \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right] \Bigg|_{x=0}^{x=\infty} \quad (9)$$

where

$$u_0 = -Q(wx) = -w^n x^n$$
$$u_1 = -L(Q(wx)) = -nw^n x^{n-1}$$
$$u_2 = -L^2(Q(wx)) = -n(n-1)w^n x^{n-2}$$
$$\vdots \quad \vdots$$
$$u_n = -L^n(Q(wx)) = -n!w^n \qquad n = 1, 2, ...$$

By definition of Sumudu transform

$$S[x^n] = \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right] \Bigg|_{x=0}^{x=\infty}$$
$$= n!w^n$$

Example 3

Let $Q(x) = e^{ax}$, then $Q(wx) = e^{awx}$, and

$$S\left[e^{ax}\right] = \int_0^\infty e^{-x} e^{awx} dx$$

$$= \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right] \Bigg|_{x=0}^{x=\infty} \tag{10}$$

where

$$u_0 = -Q(wx) = -e^{awx}$$
$$u_1 = -L\left(Q(wx)\right) = -awe^{awx}$$
$$u_2 = -L^2\left(Q(wx)\right) = -a^2w^2e^{awx}$$
$$\vdots \qquad \vdots$$
$$u_n = -L^n\left(Q(wx)\right) = -a^n w^n e^{awx}$$

By definition of Sumudu transform

$$S\left[e^{ax}\right] = \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right] \Bigg|_{x=0}^{x=\infty}$$
$$= 1 + aw + (aw)^2 + \ldots$$
$$= \frac{1}{1 - aw}$$

Example 4

Let $Q(x) = \sin(ax)$, then $Q(wx) = \sin(awx)$, and

$$S\left[\sin(ax)\right] = \int_0^\infty e^{-x} \sin(awx) dx$$

$$= \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right] \Bigg|_{x=0}^{x=\infty} \tag{11}$$

where

$$u_0 = -Q(wx) = -\sin(awx)$$
$$u_1 = -L\left(Q(wx)\right) = -aw\cos(awx)$$
$$u_2 = -L^2\left(Q(wx)\right) = a^2w^2\sin(awx)$$
$$\vdots \qquad \vdots$$
$$u_n = -L^n\left(Q(wx)\right) = -a^n w^n \sin(awx + n\pi/2) \qquad n = 1, 2, \ldots$$

By definition of Sumudu transform

$$S\left[\sin(ax)\right] = \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right] \Bigg|_{x=0}^{x=\infty}$$
$$= aw\left\{1 - (aw)^2 + (aw)^4 - (aw)^6 + (aw)^8 \ldots\right\}$$
$$= \frac{aw}{1 + (aw)^2}$$

Similarly, it can be shown that

$$S\left[\cos(ax)\right] = \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right] \Bigg|_{x=0}^{x=\infty}$$
$$= \frac{1}{1 + (aw)^2}$$

Example 5

Let $Q(x) = \sinh(ax)$, then $Q(wx) = \sinh(awx)$, and

$$S\left[\sinh(ax)\right] = \int_0^\infty e^{-x} \sinh(awx) dx$$

$$= \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right] \Bigg|_{x=0}^{x=\infty} \tag{12}$$

where

$$u_0 = -Q(wx) = -\sinh(awx)$$
$$u_1 = -L\left(Q(wx)\right) = -aw\cosh(awx)$$
$$u_2 = -L^2\left(Q(wx)\right) = -a^2w^2\sinh(awx)$$
$$\vdots \qquad \vdots$$
$$u_n = -L^n\left(Q(wx)\right) = -(-i)^n \left(aw\right)^n \sinh\left(awx + \frac{n\pi}{2}\right) \qquad n = 1, 2, \ldots$$

By definition of Sumudu transform

$$S\left[\sinh(ax)\right] = \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right] \Bigg|_{x=0}^{x=\infty}$$
$$= aw\left\{1 + (aw)^2 + (aw)^4 + (aw)^6 + (aw)^8 \ldots\right\}$$
$$= \frac{aw}{1 - (aw)^2}$$

Similarly, it can be shown that

$$S\left[\cosh(ax)\right] = \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right] \Bigg|_{x=0}^{x=\infty}$$
$$= \frac{1}{1-(aw)^2}$$

Example 6

Let $Q(x) = xe^{ax}$, then $Q(wx) = (wx)e^{awx}$, and

$$S\left[xe^{ax}\right] = \int_0^{\infty} e^{-x}(wx)e^{awx}dx$$
$$= \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right] \Bigg|_{x=0}^{x=\infty} \qquad (13)$$

where

$$u_0 = -Q(wx) = -(wx)e^{awx}$$
$$u_1 = -L\left(Q(wx)\right) = -\left\{aw^2xe^{awx} + we^{awx}\right\}$$
$$u_2 = -L^2\left(Q(wx)\right) = -\left\{a^2w^3xe^{awx} + 2aw^2e^{awx}\right\}$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$u_n = -L^n\left(Q(wx)\right) = -\frac{(aw)^n e^{awx}(awx+n)}{a}$$

By definition of Sumudu transform

$$S\left[xe^{ax}\right] = \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right] \Bigg|_{x=0}^{x=\infty}$$
$$= w + 2aw^2 + 3a^2w^3 + 4a^3w^4 \ldots$$
$$= w\left\{1 + 2aw + 3(aw)^2 + 4(aw)^3 + \ldots\right\}$$
$$= \frac{w}{(1-aw)^2}$$

Example 7

Let $Q(x) = BesselJ(0,x)$, then $Q(wx) = BesselJ(0,wx)$, and

$$S\left[BesselJ(0,x)\right] = \int_0^{\infty} e^{-x} BesselJ(0,wx)edx$$
$$= \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right] \Bigg|_{x=0}^{x=\infty} \qquad (14)$$

where

$$u_0 = -Q(wx) = -BesselJ(0,wx)$$
$$u_1 = -L\left(Q(wx)\right) = BesselJ(1,wx)w$$
$$u_2 = -L^2\left(Q(wx)\right) = BesselJ(0,wx) - \frac{BesselJ(1,wx)}{wx}w^2$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$u_n = -L^n\left(Q(wx)\right) = -\frac{\sum_{m=0}^{n}(-1)^m \binom{n}{m} BesselJ(-n+2m,wx)w^n}{2^n}$$

By definition of Sumudu transform

$$S\left[BesselJ(0,x)\right] = \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right] \Bigg|_{x=0}^{x=\infty}$$
$$= 1 - \frac{1}{2}w^2 + \frac{3}{8}w^4 + O(w^6)$$
$$= \frac{1}{\sqrt{1+w^2}}$$

Similarly, it can be shown that

$$S\left[BesselJ(0,2\sqrt{kx})\right] = \left[e^{-x} \sum_{n=0}^{\infty} u_n(x) \right] \Bigg|_{x=0}^{x=\infty} = e^{-kx}$$

Conclusion

Adomian decomposition method is successfully employed to compute Sumudu transform of some typical functions. The solution is found in the form of a convergent power series with easily computed components. Numerical examples are presented to show the efficiency of the method and a MAPLE code is provided that computes Sumudu transform of typical functions quickly. It is demonstrated that like HPM, this method is also quite easy and fast to compute Sumudu transforms.

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APPENDIX

Sample maple worksheet for computing sumudu transform:

```
>restart:
>Q(w*x):=BesselJ(0,w*x):
>v[0]:=-Q(w*x):
>v[1]:=diff(v[0],x):
>for j from 1 to 10 do v[j]:=diff(v[j-1],x) end do:
>G(w):=evalf(subs(x=infinity,(exp(-
x)*(sum(v[k],k=0..n))))-subs(x=0,(exp(-
x)*(sum(v[k],k=0..n)))));
```