Pakistan Journal of Applied Economics, Vol.27 No.1, (1-27), Summer, 2017

DETERMINANTS OF STOCKS FOR OPTIMAL PORTFOLIO

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Basically this is an empirical study which aims to test the Markowitz Modern portfolio theory (MPT) or the mean-variance analysis. Fund managers and general investors seek a portfolio that yields maximum return with minimum risk. The problem of investors is dual in nature, as Markowitz showed, i.e., the indifferent choice of risk and return. Though, diversification reduces non-systematic risk but due to limited resources one cannot afford to invest in all stocks, therefore it is pertinent to know that what should be the minimum level of stocks in a portfolio that produces maximum return and minimum risk. The theoretical framework of Markowitz MPT tested by computed 134 months expected the return of thirty-two stocks, thirty-one variances and 465 co-variances, in order to evaluate efficient portfolio frontier.

I. Introduction

The main theme of this study is to test the Markowitz Modern portfolio theory (MPT), or mean-variance analysis, empirically. For this purpose, necessary steps were taken to compute all elements of MPT on live data of companies listed with the Karachi Stock Exchange. Non-systematic risk can be reduced by means of diversification as return of portfolio is a weighted average return; hence consensus on risk reduction capabilities of diversification is unanimous. However, there is no agreement on the number of stocks or assets that are required to be included in a portfolio; since it may differ from one market to another and from one period to another, in the same market as attention drew by [Kisaka, et al., (2015)].

In the theoretical framework of economics, decisions are based on rational choice by keeping in view the scarcity of resources and preference. Markowitz identified that there is trade-off between risk and return or how to optimize utility within the given resources [(Kaplan 1998)]. The Markowitz model assumes that investors want to maximize their return at a given level of risk or minimize risk for required return. This is the reason due to which Markowitz model is known as the mean-variance theory [Fama and French (2004)]. In the theory of portfolio

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management, computation of weights play significant role. Therefore, when we say portfolio, it is well diversified and it means that wealth has been distributed among different assets with appropriate proportion (weights) and when we say that portfolio is 'poorly diversified' then it is meant that assets are not weighted properly. Hence, any change in individual stocks weight, variance and covariance will change the risk level [Statman (1987)]. Mathematically, it can be proved that minimum portfolio occurs with equal weights when securities have equal variance. Weights can be computed to achieve minimum variance and can be derived to give zero variance when correlation coefficient equals -1.0 [Reilly and Brown (1999)].

Investment in stock market is a risky decision since actual returns vary greatly (deviate) from its expected returns. Markowitz (1952) was the first to point out as to how an investor could reduce the standard deviation or risk specific to a particular stock (i.e., non-systematic) by selecting stocks with appropriate weights [Brealey, et al. (2011)]. Expected return or mean variance or standard deviation and covariance or coefficients of correlation are key parameter in Markowitz model which can be estimated from the historical data. The aim is to determine the optimal portfolio by using the mean-variance model [Kisaka, et al., (2015)] As weights of individual stocks that are dependent on covariance can be computed with more accuracy in the global minimum variance portfolio(GMVP); as a matter of fact, weights are dependent on covariance and not on mean [[Kan and Zhou (2007)]. For investors who wish to optimize their investment portfolio, an efficient portfolio frontier can be developed from GMVP Markowitz (1952). Bailey and Prado (2013) define the efficient frontier in their discussion on Modern Portfolio Theory (MPT) as an average excess return (over and above the risk-free rate) for any given risk level.

Alexander and Christoph (2005)] says that the concept of efficient frontier has become an integral part of modern investment theory. In this paper, the researcher replicated this modern portfolio theory of Harry Markowitz to construct a portfolio having maximum return against different risk level on the Karachi Stock Exchange (KSE-100) Zivot (2013) also supports the Harry Markowitz's modern theory of portfolio and asserts that risk return problem can be simplified by focusing on the efficient portfolio. For this purpose he developed the model in 'R' soft-ware. This study used his codes to evaluate optimal tangency portfolio on historical data of thirty-one stocks of Pakistani companies.

After the introduction (Section I), literature review is presented in Section II. The data (Section III) is divided into methodology of data selection, enumeration of descriptive statistics, finding the criteria to determine global minimum portfolio, determination efficient frontier portfolio and development of tangency portfolio to determine minimum risk locus. Finally, the data analysis is followed by conclusion in Section IV.

II. Literature Review

The first set of literature belongs to those researchers who identified as to what should be the size of portfolio. Newbould and Poon (1993), Tang (2004), Solnik (1990) wrote that 8-20 stocks are sufficient to reduce the risk by mean of diversification. Studies by Evans and Archer (1968), Fisher and Lorie (1970), and Tole (1982) exhibited that 90 per cent diversification advantage can be gained by including 12-18 stocks in a portfolio. Statman (1987) resolved that a well-diversified portfolio must contain at least 30 to 40 stocks.

Frahm and Wiechers (2011) used equally weighted 40 assets monthly return data for portfolios found that diversification effect among different assets contributed to portfolio performance. 'Gupta, et al. (2001) are of the opinion that 27 assets were required in the Malaysian stock market to form a well-diversified portfolio; while Zulkifli, et al. (2008) says that after scrutiny, benefit of diversification can be totally attained by investing in a portfolio of 15 stocks remaining in the same market.

Tsui, et al. (1983) found that a well-diversified portfolio in Singapore stock market consisted of 40 randomly selected securities. Nyariji (2001) evaluated the risk reduction benefits of portfolio diversification at the NSE and established that risk minimizing portfolio was 13 securities. One of the study by Yasmeen, et al. (2012) was conducted on validation of CAMP with reference to the Karachi Stock Exchange using the daily stock returns of the top 20 companies listed on the Karachi stock exchange for the period of 16th December 2008 to 26th February 2010 and found that during this period, companies selected for analysis have a continuous listing on KSE. The selection was made on the basis of trading volume excluding stocks that were traded irregularly or had small trading volumes. Another study conducted by Tang (2004), says that on an average, for any population of 100 stocks, 1.98 stocks are sufficient to eliminate 50 per cent of the diversification risk. There are 654 listed companies traded in the Pakistan Stock Exchange[1] out of which only 163 are trading actively.

The second set of literature describes naïve 1/N rule to compute portfolio, the Global Minimum Variance Portfolio, the Efficient Frontier of portfolios and the Tangency Portfolio. In this direction the first and foremost important element is the data and data source; and therefore, the data has been extracted from an extensively used source, the 'Business Recorder', [Husain (1998), Nazir and Nawaz (2010)]. Demiguel, et al. (2009) stated that though the naïve 1/N rule is quite easy and simple but it can accomplish its results, remarkably well under certain conditions; when the asset returns have equal means and variances, and when they are independent. DeMiguel (2007) says that under the minimum-variance strategy, they choose the portfolio of risky assets that minimizes the variance of returns and put restriction on valuation of moments of asset returns. Alexander and Christoph (2005), wrote

[1] Formally known as the Karachi Stock Exchange.

that for estimating the 'global minimum variance portfolio, it is assumed that stocks differ only with respect to their risk and have equal expected return'. Since the GMVP depends on covariance matrix of stock returns, they can be computed easily and accurately. Providing the required level of return which is usually the highest return of stocks in the portfolio and distributing wealth in a manner ;would produce the same return as of highest return of stock known as the efficient portfolio. As described by Zivot (2013) when expected return is greater than the expected return of MGVP, it is known as the efficient frontier of portfolios, whose targeted return is equal to the highest expected return among all assets under consideration. Thereafter, it is necessary to compute the tangency portfolio, as Zivot (2013) states that mutual fund of risk asset is the tangency portfolio where T-bill is a mutual fund of risk-free assets and tangency portfolio weights play role in determination mutual fund of the risky assets. To elaborate it further, Kim and Boyd (2008) described the two-fund separation theorem-state that an investor with quadratic utility can separate his/her asset allocation decisions into two steps:

a) Find the tangency portfolio of risky assets having maximum Sharpe ratio, and then
b) find the mix of Tangney portfolio and the risk-free asset, as per investor's attitude toward risk.

III. Data Selection

The daily closing prices of thirty-two stocks (from 2003-14) obtained from the website of Business Recorder is widely used by researchers such as, Hussain, et al. (1998). For risk-free rate one-year treasury bills data of the same period was obtained from the State Bank of Pakistan's website. The period covered eleven years and two months from 2003 to 2014. Stocks that are taken in the sample data represents 70 per cent of the major sectors the shares of which are traded on the Karachi Stock Market[2] and about 40 per cent of market capitalization.

After transforming this data into logarithmic monthly returns the expected returns and variances, and the co-variances were computed in order to calculate efficient portfolio frontier. The data is uploaded into R software and other different packages used, like 'get portfolio, efficient portfolio, global Min portfolio, tangency portfolio and efficient frontier', as prescribed by Zivot (2006).

1. Descriptive Statistics

The notable point is to observe the movement of two stocks, i.e., either they may move in the same direction or in the opposite direction; or they are not related to each other. From the diversification point of view and more precisely from the risk reduction point of view, low or even negative correlation is desirable for de-

[2] For the list of symbols and sectors, see, Appendices A to D.

veloping a portfolio of minimum risk. It benefits when the market fall and rise of stock and vice-versa [Campbell, et al. (2002)]. Though, not perfectly negative the pair of stocks is negatively correlated to each other.

Some negatively correlated pairs are: (ACPL-MCB), (ACPL-CLOV) , (ACPL-KHTC), (HUBC-MCB), (HUBC-CLOV), (MARI-MCB), (MARI-KHTC), (PRL-PPL), (PRL-INDU), (PRL-KAPCO), (PRL-FEROZ), (PRL-KEL), (AABS-PPL), (AABS-KOHC), (AABS-INDU), (AABS-KAPCO), (AABS-ENGRO), (AABS-NATF) (AABS-FEROZ), (HABSM-ATLH), (HABSM-KAPCO), (HABSM-NATF), (HABSM-KHTC), and others pairs can be found in Table 1.Therefore, the variance of portfolio will decrease by including these pairs into efficient portfolio. As stated by Zivot (2002), 'the variance of portfolio is a weighted average of variances of individual assets plus two times the product of portfolio weights times of covariance between the assets. If both portfolio weights are positive then a positive covariance will tend to increase the portfolio variance, because both returns tend to move in the same direction, and a negative covariance will tend to reduce the portfolio variance. Thus, finding assets with negatively correlated returns can be very beneficial when forming portfolios because risk, as measured by portfolio standard deviation, is reduced. What is perhaps surprising is that forming portfolios with positively correlated assets can also reduce risk, as long as the correlation is too large'.

Investment in the stock market is a risky decision since actual returns vary greatly (deviate) from its expected returns. Harry Markowitz was the first to point out as to how an investor could reduce the standard deviation or risk specific to a particular stock (i.e., non-systematic) by selecting stocks with appropriate weights [Brealey, et al. (2011)]. Kempf and Memmel (2005) state that finding the global minimum variance portfolio is based on the assumption that stocks have equal expected returns and differ only with respect to their risk and not with respect to expected returns. Therefore, the global minimum variance portfolio is the one which has the smallest risk. All investors with the intention to optimize trade-off between the expected return and risk of their portfolio should then combine the global minimum variance portfolio with the risk free asset. The structure of the global minimum variance portfolio rest on the covariance matrix of stock returns. Since one can compute the covariance matrix accurately, therefore, the investor's risk can be reduced by focusing on the global minimum variance portfolio.

The reason for using the global minimum variance portfolio is due to the fact that its weight is dependent on covariance and not on mean; therefore, these weights can be computed with more accuracy [Zhou, (2007)]. The naïve 1/N rule is a starting point and Table 2 depicts the risk and return[3] of portfolio consisting thirty-one different stocks. However, the investors may have two different and opposite motives for investment i.e., the motives for short sale and the motives when short-sale are not permitted [Gordon (2004)] and are similar to Levy (1983), They state that, 'probably

[3] See, summary statistics in Appendix-B.

	ACPL	HUBC	MARI	LUCK	PRL	AABS	HABSM	PPL	PSO	KOHC	OGDC	PIOC	AGIL	PSMC	INDU	ATLH	MCB	BERG	FCCL	KAPCO	ENGRO	ICI	SITC	CLOV	NATF	FEFOZ	PAKT	KHTC	GLAXO	KEL	PMRS
ACPL	1	0.32	0.43	0.51	0.3	0.12	0.11	0.21	0.45	0.01	0.23	0.44	0.36	0.03	0	0.05	-0.02	0.25	0.42	0.15	0.15	0.24	0.31	-0.03	0.17	0.06	0.22	-0.01	0.16	0.12	0.13
HUBC	0.32	1	0.41	0.43	0.05	0.03	0.13	0.49	0.54	0.1	0.32	0.08	0.34	0.18	0.27	0.01	-0.03	0.26	0.28	0.2	0.22	0.24	0.17	-0.06	0.18	0	0.14	0.04	0.12	0.13	0.12
MARI	0.43	0.41	1	0.41	0.11	0.14	0.18	0.34	0.53	0.05	0.28	0.35	0.44	0.05	0.09	0.15	-0.12	0.29	0.32	0.18	0.24	0.15	0.39	0.13	0.21	0.04	0.29	-0.05	0.24	0.22	0.14
LUCK	0.51	0.43	0.41	1	0.23	0.1	0.2	0.28	0.51	0.16	0.27	0.36	0.34	0.01	0.18	0.08	0	0.25	0.45	0.31	0.24	0.25	0.28	0.03	0.2	0.05	0.29	0.09	0.15	0.07	0.09
PRL	0.3	0.05	0.11	0.23	1	0.11	0.04	-0.03	0.27	0.08	0.17	0.35	0.14	0.16	-0.05	0.03	0	0.04	0.32	-0.01	0.05	0.24	0.22	0.04	0.09	-0.01	0.29	0	0.02	-0.16	0.04
AABS	0.12	0.03	0.14	0.1	0.11	1	0.28	-0.02	0.1	-0.07	0.14	0.07	0.22	-0.1	-0.22	0.09	0.07	0.09	0.1	-0.11	-0.07	0.04	0.07	0.14	-0.01	-0.03	0.17	0.04	0.02	0.26	0.22
HABSM	0.11	0.13	0.18	0.2	0.04	0.28	1	0.01	0.14	0.11	0.09	0.13	0.14	0	0.04	-0.01	0.07	0.14	0.11	-0.06	0.03	0.09	0.16	0.11	-0.05	0.07	0.21	-0.08	0.2	0.03	0.13
PPL	0.21	0.49	0.34	0.28	-0.03	-0.02	0.01	1	0.55	0.05	0.35	0.1	0.12	0.1	0.12	0.04	0.16	0.05	0.25	-0.1	0.24	0.17	0.16	-0.12	0.16	0.03	0.05	-0.08	0.18	0.2	0.11
PSO	0.45	0.54	0.53	0.51	0.27	0.1	0.14	0.55	1	0.13	0.4	0.33	0.35	0.12	0.11	0.05	0.03	0.19	0.4	0.12	0.35	0.25	0.42	-0.06	0.16	-0.02	0.3	-0.08	0.17	0.13	0.09
KOHC	0.01	0.1	0.05	0.16	0.08	-0.07	0.11	0.05	0.13	1	0.06	0.32	0.12	0.29	0.24	0.14	0.01	0.18	0.34	0.1	0	-0.03	0.25	0.11	0.09	0.02	0.14	-0.06	0.14	0.03	0.16
OGDC	0.23	0.32	0.28	0.27	0.17	0.14	0.09	0.35	0.4	0.06	1	0.21	0.22	0.06	0.11	0.1	0.11	0.18	0.16	0	0.16	0.05	0.3	-0.2	0.08	0	0.2	0.11	0.05	0.06	0.08
PIOC	0.44	0.08	0.35	0.36	0.35	0.07	0.13	0.1	0.33	0.32	0.21	1	0.33	0.15	0.12	0.13	0.06	0.21	0.36	0.02	0.14	0.13	0.5	-0.09	0.04	0.09	0.37	0.1	0.21	-0.02	0.17
AGIL	0.36	0.34	0.44	0.34	0.14	0.22	0.14	0.12	0.35	0.12	0.22	0.33	1	0.18	0.2	0.17	-0.03	0.38	0.18	0.08	0.07	0.14	0.38	0.27	0.22	0.08	0.41	-0.02	0.15	0.06	0.35
PSMC	0.03	0.18	0.05	0.01	0.16	-0.1	0	0.1	0.12	0.29	0.06	0.15	0.18	1	0.54	0	0.12	0.13	0.29	0.02	0.07	0.04	0.03	0	0.03	0.05	0.03	-0.06	-0.02	0.07	0.06
INDU	0	0.27	0.09	0.18	-0.05	-0.22	0.04	0.12	0.11	0.24	0.11	0.12	0.2	0.54	1	0.13	0.04	0.11	0.05	0.04	0.15	0.02	0.11	-0.05	0.01	-0.02	0.02	0.08	0.08	0.06	0.1
ATLH	0.05	0.01	0.15	0.08	0.03	0.09	-0.01	0.04	0.05	0.14	0.1	0.13	0.17	0	0.13	1	0.1	0.06	0.03	0.13	0.1	-0.11	0.17	0.21	0.03	0.05	0.13	0.12	0.15	0.11	0.13
MCB	-0.02	-0.03	-0.12	0	0	0.07	0.07	0.16	0.03	0.01	0.11	0.06	-0.03	0.12	0.04	0.1	1	0.02	0.09	-0.54	0.03	0.22	0.16	-0.04	0.01	0.03	-0.01	0.07	0.27	-0.06	-0.1
BERG	0.25	0.26	0.29	0.25	0.04	0.09	0.14	0.05	0.19	0.18	0.18	0.21	0.38	0.13	0.11	0.06	0.02	1	0.23	0.02	-0.03	0.15	0.2	0.05	0.08	0	0.21	0.08	0.02	0.13	0.22
FCCL	0.42	0.28	0.32	0.45	0.32	0.1	0.11	0.25	0.4	0.34	0.16	0.36	0.18	0.29	0.05	0.03	0.09	0.23	1	0.07	0.1	0.23	0.2	0.11	0.15	-0.03	0.18	-0.11	0.14	0.03	0.04
KAPCO	0.15	0.2	0.18	0.31	-0.01	-0.11	-0.06	-0.1	0.12	0.1	0	0.02	0.08	0.02	0.04	0.13	-0.54	0.02	0.07	1	0.04	-0.06	-0.04	-0.06	0.08	-0.03	0.11	-0.02	-0.18	0.08	0.03
ENGRO	0.15	0.22	0.24	0.24	0.05	-0.07	0.03	0.24	0.35	0	0.16	0.14	0.07	0.07	0.15	0.1	0.03	-0.03	0.1	0.04	1	0.05	0.05	-0.14	0.11	0.25	0.01	0.07	-0.01	0.01	-0.06
ICI	0.24	0.24	0.15	0.25	0.24	0.04	0.09	0.17	0.25	-0.03	0.05	0.13	0.14	0.04	0.02	-0.11	0.22	0.15	0.23	-0.06	0.05	1	0.19	-0.06	0.18	0.14	0.11	0.03	0.12	0	-0.03
SITC	0.31	0.17	0.39	0.28	0.22	0.07	0.16	0.16	0.42	0.25	0.3	0.5	0.38	0.03	0.11	0.17	0.16	0.2	0.2	-0.04	0.05	0.19	1	0	0.1	-0.11	0.41	-0.03	0.34	0.07	0.21
CLOV	-0.03	-0.06	0.13	0.03	0.04	0.14	0.11	-0.12	-0.06	0.11	-0.2	-0.09	0.27	0	-0.05	0.21	-0.04	0.05	0.11	-0.06	-0.14	-0.06	0	1	0.21	0.08	0.01	-0.05	0.33	0.06	0.19
NATF	0.17	0.18	0.21	0.2	0.09	-0.01	-0.05	0.16	0.16	0.09	0.08	0.04	0.22	0.03	0.01	0.03	0.01	0.08	0.15	0.08	0.11	0.18	0.1	0.21	1	0.1	0.05	0.08	0.12	0.08	0.04
FEFOZ	0.06	0	0.04	0.05	-0.01	-0.03	0.07	0.03	-0.02	0.02	0	0.09	0.08	0.05	-0.02	0.05	0.03	0	-0.03	-0.03	0.25	0.14	-0.11	0.08	0.1	1	0.03	0.09	0.05	-0.05	0.13
PAKT	0.22	0.14	0.29	0.29	0.29	0.17	0.21	0.05	0.3	0.14	0.2	0.37	0.41	0.03	0.02	0.13	-0.01	0.21	0.18	0.11	0.01	0.11	0.41	0.01	0.05	0.03	1	0.01	0.14	0.09	0.14
KHTC	-0.01	0.04	-0.05	0.09	0	0.04	-0.08	-0.08	-0.08	-0.06	0.11	0.1	-0.02	-0.06	0.08	0.12	0.07	0.08	-0.11	-0.02	0.07	0.03	-0.03	-0.05	0.08	0.09	0.01	1	-0.01	-0.11	0.11
GLAXO	0.16	0.12	0.24	0.15	0.02	0.02	0.2	0.18	0.17	0.14	0.05	0.21	0.15	-0.02	0.08	0.15	0.27	0.02	0.14	-0.18	-0.01	0.12	0.34	0.33	0.12	0.05	0.14	-0.01	1	0.09	-0.02
KEL	0.12	0.13	0.22	0.07	-0.16	0.26	0.03	0.2	0.13	0.03	0.06	-0.02	0.06	0.07	0.06	0.11	-0.06	0.13	0.03	0.08	0.01	0	0.07	0.06	0.08	-0.05	0.09	-0.11	0.09	1	0.16
PMRS	0.13	0.12	0.14	0.09	0.04	0.22	0.13	0.11	0.09	0.16	0.08	0.17	0.35	0.06	0.1	0.13	-0.1	0.22	0.04	0.03	-0.06	-0.03	0.21	0.19	0.04	0.13	0.14	0.11	-0.02	0.16	1

TABLE 1

Pearson Correlation Image Matrix

Note: Finding the Global Minimum Variance Portfolio.

any-one who has ever tried to derive the empirical mean-variance efficient set when short sales are permitted, will confirm that most, if not all, efficient portfolios contain some proportions of negative investment. This implies that the optimal investment strategy includes the holding of some securities for long and some securities for short period'. Tables 3 and 4 show the lowest return variance for given covariance matrix which is called the global minimum variance portfolio. One with the motive of short-selling and other when short-selling is not allowed, [Kempf and Memmel (2005)]; for example, when short-selling was allowed, the short-selling portfolio standard deviation reduced from 19.25 per cent to 12.43 per cent, whereas, negative weight stocks show that short-position is being taken. William Sharpe (1990) viewed that in a short position in which one owes an asset by borrowing it from (say) broker and return it back on an agreed date anticipating that price of this asset would decline in future. In this situation the investor has a report of negative holding. Table 3 explains that investors borrow (stock) and sell the stock; and then return the borrowed stock by buying it from the market as he/she foresees that at lower price he/she will buy the same stock which has already been sold at higher price. If the price of goods fall, the investors buy it for less than the price at which the product was sold, thus making a profit.

When we put restriction on short selling the negative weighted stocks are forced to zero position; for example, a portion of wealth, not required to be allocated for MARI, LUCK, HABSM, PSO, OGDC, PIOC, AGIL, ATLH, NATF, PMRS – which were negative weighted stocks or short-positioned stocks in the previous portfolio (see Table 3). In the theory of portfolio management, computation of weights plays a significant role. Statman (1987) pointed that any change in the individual stocks weight, variance and covariance, will change the risk level. In this case, as it is evident from Table 4, the risk level increased from 12.43 to 13.66 per cent.

TABLE 2

Equally Weighted Portfolio

Monthly Annualized		
Portfolio expected return	1.13%	13.60%
Portfolio standard deviation	5.56%	19.25%

Portfolio Weights

ACPL	HUBC	MARI	LUCK	PRL	AABS	HABSM	PPL	PSO	KOHC
0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323
OGDC	PIOC	AGIL	PSMC	INDU	ATLH	MCB	BERG	FCCL	KAPCO
0.0323	0.032	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323
ENGRO	ICI	SITC	CLOV	NATF	FEROZ	PAKT	KHTC	GLAXO	KEL
0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323	0.0323
PMRS									
0.0323									

TABLE 3

The Global Minimum Variance Portfolio (Short-Selling)

Monthly Annualized		
Portfolio expected return	0.88%	10.56%
Portfolio standard deviation	3.59%	12.43%
Portfolio sharpe ratio	4.15%	14.36%

Portfolio Weights

ACPL	HUBC	MARI	LUCK	PRL	AABS	HABSM	PPL
0.0516	0.0849	-0.0544	-0.1074	0.0452	0.2372	-0.0048	0.1276
PSO	KOHC	OGDC	PIOC	AGIL	PSMC	INDU	ATLH
-0.0376	0.0106	-0.0153	-0.0184	-0.038	0.0005	0.088	-0.02
MCB	BERG	FCCL	KAPCO	ENGRO	ICI	SITC	CLOV
0.0579	0.0248	0.0118	0.1638	0.0179	0.0297	0.1363	0.077
NATF	FEROZ	PAKT	KHTC	GLAXO	KEL	PMRS	
-0.0113	0.0797	0.0094	0.0351	0.0313	0.0085	-0.0216	

TABLE 4

The Global Minimum Variance Portfolio (No Short-Selling)

Monthly Annualized		
Portfolio expected return	1.03%	12.40%
Portfolio standard deviation	3.94%	13.66%
Portfolio sharpe ratio	7.60%	26.34%

Portfolio Weights

ACPL	HUBC	MARI	LUCK	PRL	AABS	HABSM	PPL	PSO	KOHC
0.0000	0.0463	0.0000	0.0000	0.0304	0.2175	0.0000	0.0790	0.0000	0.0088
OGDC	PIOC	AGIL	PSMC	INDU	ATLH	MCB	BERG	FCCL	KAPCO
0.0000	0.0000	0.0000	0.0337	0.0612	0.0000	0.0643	0.0000	0.0000	0.1357
ENGRO	ICI	SITC	CLOV	NATF	FEROZ	PAKT	KHTC	GLAXO	KEL
0.0000	0.0345	0.0708	0.0564	0.0000	0.0790	0.0000	0.0331	0.0362	0.0132
PMRS									
0.0000									

2. The Efficient Frontier Portfolio

Bailey and Prado (2013) defined the efficient frontier in their discussion on Modern Portfolio Theory and wrote that ‘efficient frontier is a set of portfolios that yield the highest achievable mean excess return (in excess to the risk-free rate) for any given level of risk (measured in terms of standard deviation). To solve the optimization problem, Zivot (2008) state that ‘we have to set the targeted return in the mean-variance portfolio; hence, composition of weights will change to achieve the targeted return’. In this case, the mean return of PAKT is taken as the targeted return which is 2.86 per cent per month to compute the efficient portfolio frontier.

3. The Efficient Portfolio with Short-Selling

The targeted individual stock return to efficient portfolio frontier was tested by the author and it was concluded that targeted return and portfolio return were identical - unconstrained or constrained short-selling. However, they have different levels of risk, e.g., standard deviation of efficient frontier portfolio was 7.15 per cent with 2.86 per cent return when short selling was allowed; it had standard deviation of 9.83 per cent with 2.86 per cent return when short-selling was not allowed.

TABLE 5
The Efficient Frontier Portfolio (Short-Selling)

Monthly Annualized		
Portfolio expected return	2.86%	34.30%
Portfolio standard deviation	7.15%	24.77%
Portfolio sharpe ratio	29.77%	103.13%

Portfolio Weights

ACPL	HUBC	MARI	LUCK	PRL	AABS	HABSM	PPL
0.1982	-0.1582	0.0435	0.0913	-0.1305	0.3265	-0.0297	0.1315
PSO	KOHC	OGDC	PIOC	AGIL	PSMC	INDU	ATLH
-0.3624	0.0596	0.0992	-0.0694	-0.0853	0.0667	0.1588	-0.0582
MCB	BERG	FCCL	KAPCO	ENGRO	ICI	SITC	CLOV
0.0385	-0.0572	-0.0299	0.1508	0.024	0.1522	0.1294	0.0664
NATF	FEROZ	PAKT	KHTC	GLAXO	KEL	PMRS	
0.0039	0.0099	0.2753	0.1326	-0.0300	-0.1019	-0.0456	

TABLE 6

The Efficient Frontier Portfolio (No Short-Selling)

Monthly Annualized		
Portfolio expected return	2.86%	34.30%
Portfolio standard deviation	9.83%	34.05%
Portfolio sharpe ratio	21.65%	75.01%

Portfolio Weights

ACPL	HUBC	MARI	LUCK	PRL	AABS	HABSM	PPL	PSO	KOHC
0.0000	0.0000	0.0000	0.1173	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
OGDC	PIOC	AGIL	PSMC	INDU	ATLH	MCB	BERG	FCCL	KAPCO
0.0636	0.0000	0.0000	0.0000	0.0694	0.0000	0.0000	0.0000	0.0000	0.0226
ENGRO	ICI	SITC	CLOV	NATF	FEROZ	PAKT	KHTC	GLAXO	KEL
0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.4112	0.3159	0.0000	0.0000
PMRS									
0.0000									

4. *The Tangency Portfolio*

Under the risk and reward frame work with different combination of a risky portfolio using riskless asset capital allocation line (CAL), the tangency portfolio is one that touches the efficient frontier and corresponds to the optimal risky portfolio; hence it has the highest Sharpe ratio. The mean-variance tangency portfolio is on the basis of constrains, firstly when short-selling is allowed and secondly when it is not allowed. Table 7 (Tangency Portfolio) depicts the situation when short selling is allowed and, the expected monthly return, the Standard deviation and Sharpe ratio are 9.75 per cent, 27.95 per cent and 32.29 per cent, respectively. As pointed out by Keykhaei and Jahandideh (2013) 'in order to find the tangency portfolio, it is enough to find the efficient portfolios and recognize the tangency portfolios which maximizes the Sharpe ratio'. It may be noted that Sharpe ratio of efficient portfolio with and without constrain are 29.77 per cent and 21.65 per centre respectively, which has now increased to 32.29 per cent and 22.10 per cent by constructing a tangency portfolio.

5. *Tangency Efficient Portfolio Frontier*

The highest return of 8.97 per cent per month (annual 107.64 per cent) can be obtained by selecting the portfolio 'one' provided there is a willingness to accept 25.51 per cent monthly risk (annual 88.37 per cent), ignoring the portfolio 20 that produce negative return – all others offer returns along with their associated risk so that investors may choose the best one as per their taste of risk. Table 10 depicts the Returns of 20 portfolios, assuming that the short-selling is allowed when Min Alpha = -2 and Max Alpha = 1.5.

TABLE 7

The Tangency Portfolio (Short-Selling is Allowed)

	Monthly	Annualized
Portfolio expected return	9.75%	117.10%
Portfolio standard deviation	27.95%	96.84%
Portfolio sharpe ratio	32.29%	111.84%

Portfolio Weights

ACPL	HUBC	MARI	LUCK	PRL	AABS	HABSM	PPL
0.7089	-1.0048	0.3848	0.7832	-0.7425	0.6375	-0.1164	0.1449
PSO	KOHC	OGDC	PIOC	AGIL	PSMC	INDU	ATLH
-1.4938	0.2305	0.4981	-0.2471	-0.2502	0.2976	0.4054	0.1914
MCB	BERG	FCCL	KAPCO	ENGRO	ICI	SITC	CLOV
-0.0292	-0.3428	-0.1751	0.1056	0.0451	0.5789	0.105	0.0295
NATF	FEROZ	PAKT	KHTC	GLAXO	KEL	PMRS	
0.0569	-0.2332	1.2018	0.4721	-0.2436	-0.4866	-0.1289	

TABLE 8

Tangency Portfolio (Short-Selling is not Allowed)

	Monthly	Annualized
Portfolio expected return	2.42%	29.04%
Portfolio standard deviation	7.63%	26.44%
Portfolio sharpe ratio	22.10%	76.56%

Portfolio Weights

ACPL	HUBC	MARI	LUCK	PRL	AABS	HABSM	PPL	PSO	KOHC
0.0000	0.0000	0.0000	0.0693	0.0000	0.0602	0.0000	0.0000	0.0000	0.0000
OGDC	PIOC	AGIL	PSMC	INDU	ATLH	MCB	BERG	FCCL	KAPCO
0.0612	0.0000	0.0000	0.0000	0.1196	0.0000	0.0239	0.0000	0.0000	0.0969
ENGRO	ICI	SITC	CLOV	NATF	FEROZ	PAKT	KHTC	GLAXO	KEL
0.0000	0.0561	0.0000	0.0000	0.0000	0.0000	0.2939	0.2189	0.0000	0.0000
PMRS									
0.0000									

The highest return of 3.57 per cent per month (annual 42.84 per cent) can be obtained by selecting the portfolio 20, provided there is willingness to accept 20.27 per cent monthly risk (annual 70.22 per cent); all other offers returns along with their associated risk so that investor may choose the best one as per their taste of risk. Table 10 depicts the Returns of 20 portfolios, assuming that the short-selling is not allowed when Min Alpha = -2 and Max Alpha = 1.5.

TABLE 9

Twenty Portfolios when Short-Selling is Allowed

Frontier portfolios' expected returns and standard deviations

	Port 1	**Port 2**	**Port 3**	**Port 4**	**Port 5**	**Port 6**	**Port 7**	**Port 8**	**Port 9**	**Port 10**
ER	0.0897	0.0847	0.0797	0.0748	0.0698	0.0648	0.0599	0.0549	0.0499	0.045
SD	0.2551	0.2398	0.2245	0.2092	0.1939	0.1787	0.1635	0.1484	0.1334	0.1186
	Port 11	**Port 12**	**Port 13**	**Port 14**	**Port 15**	**Port t6**	**Port 17**	**Port 18**	**Port 19**	**Port 20**
ER	0.04	0.035	0.0301	0.0251	0.0201	0.0152	0.0102	0.0052	0.0003	-0.0047
SD	0.1039	0.0895	0.0755	0.0623	0.0504	0.0411	0.0362	0.0376	0.0447	0.0553

TABLE 10

Twenty Portfolios when Short-Selling is not Allowed

Frontier portfolios' expected returns and standard deviations

	Port 1	**Port 2**	**Port 3**	**Port 4**	**Port 5**	**Port 6**	**Port 7**	**Port 8**	**Port 9**	**Port 10**
ER	0.0103	0.0116	0.013	0.0143	0.0157	0.017	0.0183	0.0197	0.021	0.0224
SD	0.0394	0.04	0.0412	0.0433	0.0461	0.0495	0.0535	0.058	0.063	0.0684
	Port 11	**Port 12**	**Port 13**	**Port 14**	**Port 15**	**Port 16**	**Port 17**	**Port 18**	**Port 19**	**Port 20**
ER	0.0237	0.025	0.0264	0.0277	0.0291	0.0304	0.0317	0.0331	0.0344	0.0357
SD	0.0742	0.0803	0.0867	0.0934	0.1011	0.1099	0.1206	0.1383	0.1673	0.2027

6. Commentary on the Tangency Efficient Portfolio Frontier

In Figure 1, the curved line depicts the efficient portfolio frontier and dots on the line show different 20 portfolios which are constructed by applying different combinations of weights in each portfolio. The straight-line is the Capital Market Line (CML) which describe all efficient portfolios making contacts with the curve of efficient frontiers return, i.e., the best portfolio with respect to the excess return/risk ratio and contain shares of every stock in proportion to the stock's weight in the market, Arratia (2014). As Keykhaei and Jahandideh (2013) pointed out that, in fact all combinations of a risky portfolio and a riskless asset can be represented by a line - Capital Allocation

Line (CAL), originating at the riskless asset and passing through the risky portfolio, in the Mean-Standard Deviation (M-SD) plane. There exists a CAL termed by optimal CAL, which dominates the other CALs. When borrowing of riskless asset is allowed, the efficient frontier is the optimal CAL. The optimal CAL has the highest possible slope and is tangent to the efficient frontier of risky assets; thus, the risky portfolio corresponding to the tangent point by tangency portfolio is denoted. Indeed, tangency portfolio is the efficient portfolio which maximizes the famous Sharpe ratio. The large green dot shows the global minimum variance portfolio or a portfolio that has smallest risk. The Tangency dot shows that at this point there is optimal return with minimum risk. All points below this points have low return and low risk, whereas, all portfolios above this point have greater return but with high risk. If there is a restriction on short-selling, the risk-free rate is = 0.088.In this case the efficient frontier will comprise for these nine stocks: LUCK, AABS, OGDC, INDU, MCB, KAPCO, ICI, PAKT and KHTC, the weight ofwhich = 1 and monthly return = 2.42 per cent (annual return = 29.04 per cent). On one hand, if short-selling is allowed the return would increase by many fold, i.e., 9.75 per cent monthly or 117.10 per centannually (Table 8).

If the expected return on resulting portfolio is greater than the expected return on global minimum variance portfolio then it is an efficient frontier portfolio; else it is inefficient frontier portfolio. Therefore, the tangency portfolio is an efficient portfolio that touches the CML, above the global minimum variance portfolio.

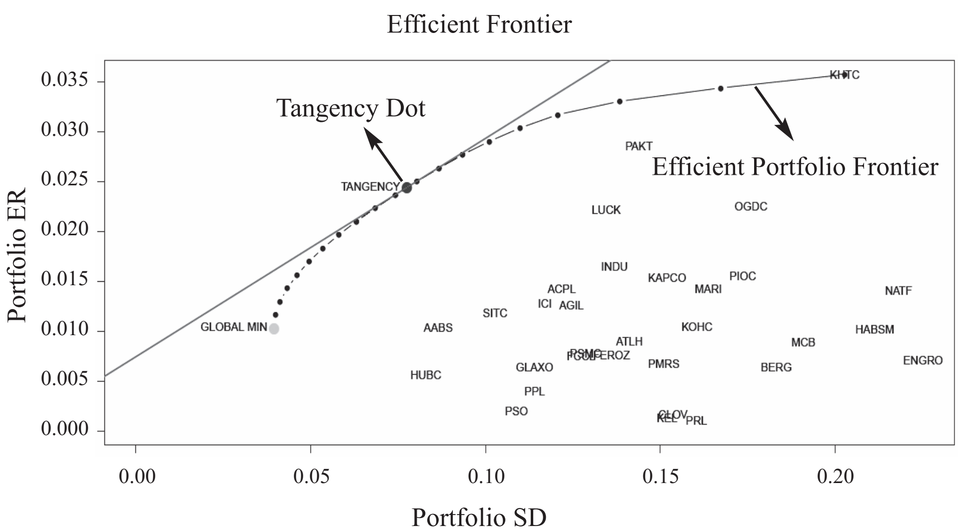

FIGURE 1

Tangency Efficient Portfolio Frontier
(min.Alpha = -2 and max.Alpha = 1.5 constrained)

IV. Conclusion

According to one estimate, the average rate of return on KSE-100 index is 18 per cent [Naqvi (2014)], whereas, the present study annualized the expected rate of return of tangency portfolio when the short selling is not allowed (consist on only nine stocks); 29.04 per cent which give 1.614 times greater return than to KSE-100 stocks index. The portfolio selection problem was annualized by the mean-risk approach and found some practical implication, such as mentioned in its statistical financial highlight report of 2011. The Mutual Fund Association of Pakistan reported the combined average return of 119 open-ended funds year 2002-2011whichis 31 per cent i.e., close to 29 per cent return of one fund that are formed in this paper, whereas, combined 16 close-ended average return is 23 per cent. Therefore, it is pertinent to conclude that:

1. Performance (judged by ranked Sharpe ratio and return) of fund management companies will improve significantly, by adopting mean-variance analysis approach as discussed (throughout) in this paper.

2. Not only the performance can be improved but also it can reduce the number of shares in a portfolio that produce maximum return and with minimum risk. As stated in this study, it is started to estimate the mean-variance of the portfolio consisting on thirty-two stocks and ended with the tangency portfolio (when short-selling is not allowed) having only nine stocks. Practically, speaking on this point, it is very important to note that as Assets Management Companies (AMC)in Pakistan allocates big-chunk of their sources in equity (Equity Fund) and then reallocate these sources into different assets. This study recommends that if these AMCs adopt the whole process they can achieve the targeted risk and return mix.

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APPENDIX-A

Mathematics of Efficient Portfolio

The steps for constructing efficient frontier are:

1. Compute the global minimum variance portfolio m by solving. and compute $\mu_{p,m} = m'\mu$ *and* $\sigma^2_{p,m} = m'\sum m$.
2. Compute the efficient portfolio x by target expected return equal to the maximum expected return of assets under consideration,
   i.e., solve, with $\mu_0 = max\{\mu_1, \mu_2, \mu_3\}$, and compute $\mu_{p,x} = x'\mu$ and $\sigma^2_{p,m} = x'\sum x$.
3. Compute $cov(R_{p,m}, R_{p,x}) = \sigma_{mx} = m'\sum x$.
4. Create an initial grid of α values {1,0. 9..., −0.9, −1}; compute the frontier portfolios z using (1.29), and compute their expected returns and variances using and, respectively.
5. Plot $\mu_{p,z}$ against $\sigma_{p,z}$ and adjust the grid of α value to create a nice plot.

Explanation of Step 1

The return on the portfolio using matrix notation is

$$R_{p,x} = x'R = (x_A, x_B, x_C) \cdot \begin{pmatrix} R_A \\ R_B \\ R_C \end{pmatrix} = x_A R_A + x_B R_B + x_C R_C$$

Similarly, the expected return on the portfolio is

$$\mu_{p,x} = E[x'R] = x'E[R] = x'\mu = (x_A, x_B, x_C) \begin{pmatrix} \mu_A \\ \mu_B \\ \mu_C \end{pmatrix} = x_A\mu_A + x_B\mu_B + x_C\mu_C$$

The variance of the portfolio is

$$\sigma^2_{p,x} = var(x'R) = x'\sum x = (x_A, x_B, x_C) \cdot \begin{pmatrix} \sigma^2_A & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma^2_B & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma^2_C \end{pmatrix} \begin{pmatrix} x_A \\ x_B \\ x_C \end{pmatrix}$$

$$= x^2_A \sigma^2_A + x^2_B \sigma^2_B + x^2_C \sigma^2_C + 2x_A x_B \sigma_{AB} + 2x_A x_C \sigma_{AC} + 2x_B x_C \sigma_{BC}$$

The condition that the portfolio weights sum to one can be expressed as

$$x'1 = (x_A, x_B, x_C) \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = x_A + x_B + x_C = 1$$

where 1 is a 3 × 1 vector with each element equal to 1.

Consider another portfolio with weights $y = (y_A, y_B, y_C)'$. The return on this portfolio is

$$R_{p,y} = y' R = y_A R_A + y_B R_B + y_C R_C$$

Later on we will need to compute the covariance between the return on portfolio x and the return on portfolio y, cov $(R_{p,x}, R_{p,w})$. Using matrix algebra, this covariance can be computed as

$$\sigma_{xy} = \text{cov}\,(R_{p,x}, R_{p,y}) = \text{cov}\,(x' R, y' R) =$$

$$x' \Sigma y = (x_A, x_B, x_C) \cdot \begin{pmatrix} \sigma_A^2 & \sigma_{AB} & \sigma_{AC} \\ \sigma_{AB} & \sigma_B^2 & \sigma_{BC} \\ \sigma_{AC} & \sigma_{BC} & \sigma_C^2 \end{pmatrix} \begin{pmatrix} y_A \\ y_B \\ y_C \end{pmatrix} =$$

$$x_A y_A \sigma_A^2 + x_B y_B \sigma_B^2 + x_C y_C \sigma_C^2 + (x_A y_B + x_B y_A)\, \sigma_{AB} + (x_A y_C + x_C y_A)\, \sigma_{AC} + (x_B y_C + x_C y_B) \sigma_{AC}$$

The global minimum variance portfolio m=(m_A,m_B,m_C) for the three asset case solves the constrained minimization problem

$$\min_{m_A, m_B, m_C} \sigma_{p,m}^2 = m_A^2 \sigma_A^2 + m_B^2 \sigma_B^2 + m_C^2 \sigma_C^2 + 2 m_A m_B \sigma_{AB} + 2 m_B m_C \sigma_{BC}$$

$$\text{s.t. } m_A + m_B + m_C = 1$$

The Lagrangian for this problem is

$$L(m_A, m_B, m_C, \lambda) = m_A^2 \sigma_A^2 + m_B^2 \sigma_B^2 + m_C^2 \sigma_C^2 + m_A m_B \sigma_{AB} + 2 m_A m_C \sigma_{AC} + 2 m_B m_C \sigma_{BC} + \lambda\, (m_A + m_B + m_C - 1.$$

and the first order conditions (FOCs) for a minimum are

$$0 = \frac{\partial L}{\partial m_A} = 2 m_A \sigma_A^2 + 2 m_B \sigma_{AB} + 2 m_C \sigma_{AB} + \lambda$$

$$0 = \frac{\partial L}{\partial m_A} = 2 m_B \sigma_B^2 + 2 m_A \sigma_{AB} + 2 m_C \sigma_{BC} + \lambda$$

$$0 = \frac{\partial L}{\partial m_A} = 2 m_C \sigma_C^2 + 2 m_A \sigma_{AC} + 2 m_B \sigma_{BC} + \lambda$$

$$0 = \frac{\partial L}{\partial \lambda} = m_A + m_B + m_C - 1$$

The FOCs (1.5) gives four linear equations in the four unknown which can be solved to find the global minimum variance portfolio weights m_A, m_B and m_C. Using matrix notation, the problem (1.4) can be concisely expressed as

$$\min_m \sigma^2_{p,m} = m \sum m' \; s.t. \; m' 1 = 1$$

Explanation of Step 2

Following Markowitz,it is assumed that investors wish to find the portfolios that have the best expected return-risk trade-off.Markowitz characterized these effcient portfolios in two equivalent ways: first, the investors seek to find the portfolios that maximize portfolio expected return for a given level of risk as measured by portfolio variance. Let $\sigma^2_{p,0}$ denote a target level of risk. Then Harry Markowitz characterized the constrained maximization problem to find an effcient portfolio as:

$$\max_x \mu_p = x'\mu \; s.t.$$
$$\sigma^2_p = x'\sum x = \sigma^2_{p,0} \text{ and } x' 1 = 1$$

Markowitz showed that investor's problem of maximizing portfolio expected return subject to a target level of riskwhich has an equivalent dual representation in which the investor minimizes the risk of portfolio (as measured by the portfolio variance) subject to a target expected return level. Let $\mu_{p,0}$ denote a target expected return level. Then the dual problem is the constrained minimization problem

$$\max_x \sigma^2_{p,x} = x'\sum x \; s.t.$$
$$\mu_p = x'\mu = \sigma_{p,0,} \text{ and } x' 1 = 1$$

Creating a Frontier Portfolio from Two Effcient Portfolios-Step-Wise Explanation of Steps 3 and 4

Let $x = (x_A, x_B, x_C)$ and $y = (y_A, y_B, y_C)$ be any two minimum variance portfolios with different target expected returns $x\mu = \mu_{p,0} \neq y\mu = \mu_{p,1}$ That is, portfolio x solves

$$\max_x \sigma^2_{p,x} = x'\sum x \;\; \text{s.t.} \;\; x'\mu \; = \; \mu_{p,0,} \text{ and } x' 1 = 1,$$

and portfolio y solves

$$\max_y \sigma^2_{p,y} = x'\sum y \;\; \text{s.t.} \;\; y'\mu \; = \; \mu_{p,1,} \text{ and } y' 1 = 1.$$

Let α be any constant and define the portfolio *z* as a linear combination of portfolios *x* and *y*:

$$z = \alpha \,.\, x + (1 - \alpha) \,.\, y$$

$$= \begin{pmatrix} \alpha x_A + (1 - \alpha) y_A \\ \alpha x_B + (1 - \alpha) y_B \\ \alpha x_C + (1 - \alpha) y_C \end{pmatrix}$$

The efficient frontier of portfolios, i.e., those frontier portfolios with expected return are greater than the expected return on global minimum variance portfolio, which can be conveniently created using, (1.29) with two specific efficient portfolios. The first efficient portfolio is the global minimum variance portfolio (1.4). The second is the efficient portfolio whose target expected return is equal to the highest expected return among all the assets under consideration.
http://faculty.washington.edu/ezivot/econ424/portfolioTheoryMatrix.pdf
Eric Zivot, August 7, 2013, University of Washington.

APPENDIX-B

Summary Statistics

Symbols	Nobs	Min	Max	1 Quartile	3 Quartile	Mean	Median	Sum	SE Mean	LCL Mean	UCL Mean	Variance	St.dev	Skewness	Kurtosis
ACPL	134	0.426	0.376	-0.060	0.088	0.014	0.013	1.915	0.011	-0.007	0.035	0.015	0.122	-0.081	1.620
HUBC	134	-0.421	0.247	-0.028	0.051	0.006	0.008	0.759	0.007	-0.009	0.020	0.007	0.083	-1.108	4.900
MARI	134	-0.549	0.485	-0.053	0.078	0.014	0.004	1.916	0.014	-0.014	0.042	0.027	0.164	0.060	1.594
LUCK	134	-0.614	0.468	-0.033	0.093	0.022	0.025	2.980	0.012	-0.001	0.045	0.018	0.135	-0.581	3.820
PRL	134	-0.579	0.455	-0.083	0.075	0.001	-0.001	0.141	0.014	-0.026	0.028	0.026	0.160	-0.067	1.595
AABS	134	-0.184	0.458	-0.035	0.046	0.010	0.000	1.389	0.008	-0.004	0.025	0.008	0.087	1.795	6.847
HABSM	134	-1.048	1.170	-0.046	0.086	0.010	0.018	1.367	0.018	-0.026	0.046	0.045	0.211	-0.323	13.633
PPL	134	-0.508	0.607	-0.038	0.038	0.004	0.004	0.539	0.010	-0.016	0.024	0.013	0.114	0.337	7.891
PSO	134	-0.615	0.380	-0.042	0.056	0.002	0.002	0.270	0.009	-0.017	0.021	0.012	0.109	-0.956	7.936
KOHC	134	-0.687	0.502	-0.079	0.095	0.011	0.007	1.401	0.014	-0.017	0.038	0.026	0.161	-0.380	2.753
OGDC	134	-0.636	1.682	-0.034	0.057	0.023	0.009	3.025	0.015	-0.008	0.053	0.031	0.176	5.899	57.794
PIOC	134	-0.913	0.488	-0.086	0.108	0.016	0.010	2.084	0.015	0.014	0.045	0.030	0.174	-0.859	5.170
AGIL	134	-0.483	0.363	-0.054	0.081	0.013	-0.003	1.696	0.011	-0.009	0.034	0.016	0.125	-0.088	1.874
PSMC	134	-0.414	0.398	-0.056	0.073	0.008	0.004	1.041	0.011	-0.014	0.030	0.017	0.129	0.038	1.445
INDU	134	-0.666	0.502	-0.051	0.090	0.017	0.019	2.211	0.012	-0.007	0.040	0.019	0.137	-0.522	4.877
ATLH	134	-0.544	0.588	-0.041	0.078	0.009	0.016	1.211	0.012	-0.015	0.033	0.020	0.141	-0.586	3.900
MCB	134	-1.298	0.404	-0.039	0.094	0.009	0.015	1.197	0.017	-0.024	0.042	0.036	0.191	-3.189	17.683
BERG	134	-0.727	0.722	-0.067	0.082	0.006	0.000	0.854	0.016	-0.025	0.038	0.034	0.183	-0.060	3.688
FCCL	134	-0.384	0.351	-0.075	0.082	0.008	0.008	1.011	0.011	-0.014	0.029	0.016	0.127	-0.100	0.188
KAPCO	134	-0.272	1.562	-0.022	0.039	0.015	0.001	2.066	0.013	-0.011	0.041	0.023	0.152	7.759	77.575
ENGRO	134	-1.015	1.290	-0.052	0.082	0.007	0.007	0.950	0.019	-0.031	0.046	0.051	0.225	0.623	14.944
ICI	134	-0.643	0.325	-0.048	0.060	0.013	0.007	1.712	0.010	-0.007	0.033	0.014	0.117	-0.838	6.568
SITC	134	-0.461	0.311	-0.049	0.054	0.012	0.002	1.583	0.009	-0.006	0.029	0.011	0.103	-0.062	3.003
CLOV	134	-0.539	0.512	-0.059	0.069	0.002	0.000	0.221	0.013	-0.025	0.028	0.024	0.154	0.200	2.538
NATF	134	-1.793	0.467	-0.060	0.114	0.014	0.004	1.888	0.019	-0.023	0.051	0.048	0.218	-4.200	33.390
FEROZ	134	-0.718	0.562	-0.043	0.055	0.008	0.017	1.022	0.012	-0.016	0.031	0.019	0.136	-1.390	10.327
PAKT	134	-0.651	0.484	-0.056	0.099	0.029	0.013	3.830	0.012	0.004	0.053	0.021	0.144	0.070	4.002
KHTC	134	-0.515	0.811	-0.011	0.093	0.036	0.000	4.790	0.018	0.001	0.070	0.041	0.203	0.985	3.089
GLAXO	134	-0.427	0.439	-0.046	0.067	0.006	0.000	0.854	0.010	-0.013	0.026	0.013	0.114	-0.295	3.054
KEL	134	-0.369	0.539	-0.083	0.058	0.001	0.000	0.174	0.013	-0.025	0.027	0.023	0.152	1.115	2.912
PMRS	134	-0.409	0.574	-0.076	0.047	0.007	-0.012	0.910	0.013	-0.019	0.033	0.023	0.151	1.138	2.456

APPENDIX-C

Sr.#	Symbol	Company	Sector
1.	ACPL	Attock Cement Pakistan Ltd.	Cement
2.	AGIL	Agriautos Industries Co. Ltd.	Automobile Assembler
3.	ATLH	Atlas Honda Limited.	Automobile Assembler
4.	BERG	Berger Paints Pakistan Ltd.	Chemicals
5.	BOP	Bank of Punjab.	Commercial Bank
6.	CLOV	Clover Pakistan Limited.	Food & Personal Care Products
7.	DAWH	Dawood Hercules Corpn. Ltd	Fertilizer
8.	DGKC	D. G. Khan Cement Co. Ltd.	Cement
9.	ENGRO	Engro Corporation Limited.	Fertilizer
10.	FCCL	Fauji Cement Co. Ltd.	Cement
11.	FEROZ	Ferozsons Laboratories Ltd.	Pharmaceuticals
12.	HABSM	Habib Sugar Mills Ltd.	Sugar & Allied Industries
13.	HUBC	Hub Power Company Limited.	Power Generation & Distribution
14.	ICI	ICI Pakistan Limited.	Chemicals
15.	INDU	Indus Motor Company Ltd.	Automobile Assembler
16.	KAPCO	Kot Addu Power Company.	Power Generation & Distribution
17.	KHTC	Khyber Tobacco Co. Ltd.	Tobacco
18.	KOHC	Kohat Cement Co. Ltd.	Cement
19.	LUCK	Lucky Cement Limited.	Cement
20.	MARI	Mari Petroleum Company Ltd.	Oil & Gas Exploration Companies
21.	MCB	MCB Bank Limited.	Commercial Bank
22.	NATF	National Foods Ltd.	Food & Personal Care Products
23.	OGDC	Oil & Gas Development Co. Ltd.	Oil & Gas Exploration Companies
24.	PAKT	Pakistan Tobacco Co. Ltd.	Tobacco
25.	PIOC	Pioneer Cement Ltd.	Cement
26.	PPL	Pakistan Petroleum Limited.	Oil & Gas Exploration Companies
27.	PRL	Pakistan Refinery Ltd.	Refinery
28.	PSMC	Pak Suzuki Motors Co. Ltd.	Automobile Assembler
29.	PSO	Pakistan State Oil Co. Ltd.	Oil & Gas Marketing Companies
30.	SHEL	Shell Pakistan Ltd.	Oil & Gas Marketing Companies
31.	SITC	Sitara Chemicals Ind Ltd.	Chemicals
32.	AABS	AL-Abbas Sugar Mills Ltd.	Sugar & Allied Industries

APPENDIX-D

R-Scripts for Determinants of Stocks for Optimal Portfolio

```
ef.df<- read.table("C:/Users/Zakir Abbas/Desktop/revisedData.csv",
  header=TRUE, sep=",", na.strings="NA", dec=".", strip.white=TRUE)
source("E:\\R-FILES\\1portfolio.r").
asset.names = c("ACPL", "HUBC", "MARI", "LUCK","PRL", "AABS",
"HABSM", "PPL","PSO", "KOHC", "OGDC", "PIOC", "AGIL", "PSMC",
"INDU", "ATLH", "MCB", "BERG", "FCCL", "KAPCO","ENGRO","ICI",
"SITC", "CLOV", "NATF", "FEROZ", "PAKT", "KHTC", "GLAXO","KEL",
"PMRS")
er<- c(mean(ef.df$ACPL), mean(ef.df$HUBC), mean(ef.df$MARI),
mean(ef.df$LUCK), mean(ef.df$PRL), mean(ef.df$AABS), mean(ef.df$HABSM),
mean(ef.df$PPL),mean(ef.df$PSO), mean(ef.df$KOHC), mean(ef.df$OGDC),
mean(ef.df$PIOC), mean(ef.df$AGIL), mean(ef.df$PSMC), mean(ef.df$INDU),
mean(ef.df$ATLH), mean(ef.df$MCB), mean(ef.df$BERG), mean(ef.df$FCCL),
mean(ef.df$KAPCO), mean(ef.df$ENGRO), mean(ef.df$ICI), mean(ef.df$SITC),
mean(ef.df$CLOV), mean(ef.df$NATF), mean(ef.df$FEROZ), mean(ef.df$PAKT),
mean(ef.df$KHTC), mean(ef.df$GLAXO), mean(ef.df$KEL),
mean(ef.df$PMRS))
names(er) = asset.names
covmat<- cov(ef.df)
r.free = 0.0073
dimnames(covmat) = list(asset.names, asset.names)
er
covmat
r.free
library(fBasics)
round(basicStats(ef.df), 4)
ew = rep(1,31)/31
equalWeight.portfolio = getPortfolio(er=er,cov.mat=covmat,weights=ew)
class(equalWeight.portfolio)
names(equalWeight.portfolio)
equalWeight.portfolio
plot(equalWeight.portfolio, col= "green")
gmin.port<- globalMin.portfolio(er, covmat)
attributes(gmin.port)
print(gmin.port)
summary(gmin.port, risk.free=r.free)
plot(gmin.port, col="blue")
```

```
gmin.port.ns = globalMin.portfolio(er, covmat, shorts=FALSE)
attributes(gmin.port.ns)
print(gmin.port.ns)
summary(gmin.port.ns, risk.free=r.free)
plot(gmin.port.ns, col="blue")
target.return<- er["KHTC"]
e.port.KHTC<- efficient.portfolio(er, covmat, target.return)
e.port.KHTC
summary(e.port.KHTC, risk.free=r.free)
plot(e.port.KHTC, col="blue")
target.return = er["KHTC"]
e.port.KHTC.ns = efficient.portfolio(er, covmat, target.return, shorts=FALSE)
e.port.KHTC.ns
summary(e.port.KHTC.ns, risk.free=r.free)
plot(e.port.KHTC.ns, col="blue")
tan.port<- tangency.portfolio(er, covmat, r.free)
tan.port
summary(tan.port, risk.free=r.free)
plot(tan.port, col="blue")
tan.port.ns<- tangency.portfolio(er, covmat, r.free, shorts=FALSE)
tan.port.ns
summary(tan.port.ns, risk.free=r.free)
plot(tan.port.ns, col="blue")
ef<- efficient.frontier(er, covmat, alpha.min=-2,alpha.max=1.5, nport=20)
attributes(ef)
ef
plot(ef)
plot(ef, plot.assets=TRUE, col="blue", pch=16)
points(gmin.port$sd, gmin.port$er, col="green", pch=16, cex=2)
points(tan.port$sd, tan.port$er, col="red", pch=16, cex=2)
text(gmin.port$sd, gmin.port$er, labels="GLOBAL MIN", pos=2)
text(tan.port$sd, tan.port$er, labels="TANGENCY", pos=2)
sr.tan = (tan.port$er - r.free)/tan.port$sd
abline(a=r.free, b=sr.tan, col="green", lwd=2)
sd.vals = sqrt(diag(covmat))
mu.vals = er
plot(ef$sd, ef$er, ylim=c(0, max(ef$er)), xlim=c(0, max(ef$sd)),
xlab="portfolio sd", ylab="portfolio er", main="Efficient Portfolios")
text(sd.vals, mu.vals, labels=names(mu.vals))
abline(a=r.free, b=sr.tan)
ef.ns<- efficient.frontier(er, covmat, alpha.min=0, alpha.max=1, nport=20,
```

```
shorts=FALSE)
attributes(ef.ns)
ef.ns
summary(ef.ns)
plot(ef.ns)
plot(ef.ns, plot.assets=TRUE, col="blue", pch=16)
points(gmin.port.ns$sd, gmin.port.ns$er, col="green", pch=16, cex=2)
points(tan.port.ns$sd, tan.port.ns$er, col="red", pch=16, cex=2)
text(gmin.port.ns$sd, gmin.port.ns$er, labels="GLOBAL MIN", pos=2)
text(tan.port.ns$sd, tan.port.ns$er, labels="TANGENCY", pos=2)
sr.tan.ns = (tan.port.ns$er - r.free)/tan.port.ns$sd
abline(a=r.free, b=sr.tan.ns, col="green", lwd=2)
sd.vals = sqrt(diag(covmat))
mu.vals = er
plot(ef.ns$sd, ef.ns$er, ylim=c(0, max(ef.ns$er)), xlim=c(0, max(ef.ns$sd)),
xlab="portfolio sd", ylab="portfolio er", main="Efficient Portfolios")
text(sd.vals, mu.vals, labels=names(mu.vals))
abline(a=r.free, b=sr.tan.ns)
histPlot<- function(x, ...) {
X = as.vector(x)
H = hist(x = X, ...)
box()
grid()
abline(h = 0, col = "grey")
mean = mean(X)
sd = sd(X)
xlim = range(H$breaks)
s = seq(xlim[1], xlim[2], length = 201)
lines(s, dnorm(s, mean, sd), lwd = 2, col = "brown")
abline(v = mean, lwd = 2, col = "orange")
Text = paste("Mean:", signif(mean, 3))
mtext(Text, side = 4, adj = 0, col = "darkgrey", cex = 0.7)
rug(X, ticksize = 0.01, quiet = TRUE)
invisible(s)}
par(mfrow = c(4, 4))
main = colnames(ef.df)
for (i in 25:31) histPlot(ef.df[, i], main = main[i], col = "steelblue",
border = "white", nclass = 25, freq = FALSE, xlab = "Returns")
cor(ef.df)
R = as.matrix(ef.df)
n = ncol(R)
```

```
Names = abbreviate(colnames(R), 4)
corr<- cor(R)
ncolors<- 10 * length(unique(as.vector(corr)))
k <- round(ncolors/2)
r <- c(rep(0, k), seq(0, 1, length = k))
g <- c(rev(seq(0, 1, length = k)), rep(0, k))
b <- rep(0, 2 * k)
corrColorMatrix<- (rgb(r, g, b))
image(x = 1:n, y = 1:n, z = corr[, n:1], col = corrColorMatrix,
axes = FALSE, main = "", xlab = "", ylab = "")
axis(2, at = n:1, labels = colnames(R), las = 2)
axis(1, at = 1:n, labels = colnames(R), las = 2)
title(main = "Pearson Correlation Image Matrix")
box()
x = y = 1:n
nx = ny = length(y)
xoy = cbind(rep(x, ny), as.vector(matrix(y, nx, ny, byrow = TRUE)))
coord = matrix(xoy, nx * ny, 2, byrow = FALSE) X = t(corr)
for (i in 1:(n * n)) {
text(coord[i, 1], coord[n * n + 1 - i, 2], round(X[coord[i,
1], coord[i, 2]], digits = 2), col = "white", cex = 0.7)}
```