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A Different Approach by System of Differential Equations for the Characterization Position Vector of Spacelike Curves

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Abstract.: The purpose of this study is to obtain a characterization of unit speed spacelike curve with constant curvature and torsion in the Minkowski 3-space. According to this purpose, the position vector of a spacelike curve is expressed by a linear combination of its Serret Frenet Frame with differentiable functions. Since a spacelike curve has different kinds of frames, then we investigate the curve with respect to the Lorentzian casual characterizations of the frame. Hence we examine the results in three different cases including different subcases. Moreover, we illustrate some examples for each case.

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1. Introduction

The geometric structure of the curves can be discussed in two ways. One, which may be named as classical differential geometry, started with the beginning of calculus. In the most general sense, the classical differential geometry is the study of local properties of the curve. By local properties, we mean those properties which depend only on the behavior of the curve in a neighborhood of a point. The other aspect is called global differential geometry. Here one studies the influence of the local properties on the behavior of the entire curve. According to both aspects, we need use the curvature function κ and torsion function τ to describe the behavior of the curve. Physically, we can think of a space curve as being obtained from a straight line by bending (curvature) and twisting (torsion) [7].

There are many studies on the characterization of curves using the curvature and torsion function in different spaces. In the studies of [9, 14], constant ratio curves in Euclidean spaces and some of their characterizations are expressed. Furthermore, the definition of constant ratio curve is given in the lower manifolds of Euclidean space in [2] and the Riemannian surfaces are discussed in [3]. In addition, [4] studied the relationship between rectifying curves and twisted curves in Euclidean space. As a continuation of this work, some geometrical properties of rectifying curves are given [5]. In addition in study [1], rectifying, normal and osculating curves were studied in three-dimensional compact Lie groups.

Among the current studies, the most striking ones are the studies on the characterization of twisted curves. If the curvature and torsion functions of the curve α are different from zero, the curve α is called a twisted curve. In the study [15], it is stated that each twisted curve can be given in the following form

$$\alpha\,(s) = m_0\,(s)\,T\,(s) + m_1\,(s)\,N\,(s) + m_2\,(s)\,B\,(s)$$

where $m_0,\ m_1, m_2 : I \to \mathbb{R}$ are differentiable functions. Moreover, the characterizations of W curves are investigated in [6]. The curve α is called a W curve, if its curvature and torsion functions are constant . The simplest examples of W curves are circles, hyperbolas as planar W curves and helices as non-planar W curves. Spacelike W curves in the Minkowski 3-space are classified by Walrave in [17]. At the same time Walrave gave the relations between the curvature and torsion of the W curves in Minkowski space. Moreover, W curves in the Minkowski 3-space are investigated in [10, 11, 13].

The main purpose of this study is to examine unit speed spacelike curve with constant curvature and torsion in the Minkowski 3-space. For this purpose, the position vector of a spacelike curve is expressed by a linear combination of its Serret Frenet Frame with differentiable functions. Since a principal normal vector field of spacelike curve can be spacelike, timelike or null, then we investigate the curve in three different cases. There exist also some different subcases depending on the values of curvature and torsion of the curve. Furthermore, we give some examples to explain the results for each case.

2. PRELIMINARIES

Minkowski 3-space is the Euclidean space provided with Lorentzian product

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle_L = -u_1 v_1 + u_2 v_2 + u_3 v_3$$

where $\overrightarrow{u} = (u_1, u_2, u_3)$, $\overrightarrow{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$. By definition, this product is not positively defined. Instead, this product classifies the vectors in $\mathbb{E}_1^3$ as follows:

i)If $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_L > 0$ or $(\overrightarrow{u} = 0)$ then $\overrightarrow{u}$ is called a spacelike vector

ii) If $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_L < 0$ then $\overrightarrow{u}$ is called a timelike vector

iii) If $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_L = 0$ and $\overrightarrow{u} \neq 0$ then $\overrightarrow{u}$ is called a lightlike (or null) vector.

For each $\overrightarrow{u} \in \mathbb{E}_1^3$, the norm of $\overrightarrow{u}$ vector is defined

$$\|\overrightarrow{u}\| = \sqrt{|\langle \overrightarrow{u}, \overrightarrow{u} \rangle_L|}.$$

If $\langle \overrightarrow{u}, \overrightarrow{v} \rangle_L = 0$ then $\overrightarrow{u}$ and $\overrightarrow{v}$ vectors are said to be orthogonal. For each $\overrightarrow{u}$, $\overrightarrow{v} \in \mathbb{E}_1^3$, we may write

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle_L = -u_1 v_1 + u_2 v_2 + u_3 v_3 = u^T I^* v$$

where

$$I^* = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $\alpha : I \to \mathbb{E}_1^3$ be a regular curve. If the vector $\alpha'(s)$ is a timelike vector $\forall s \in I$, then α is called timelike curve. If α is a timelike curve and $\left\langle \alpha'(s), \alpha'(s) \right\rangle_L = -1$, then α is called unit speed timelike curve. If the vector $\alpha'(s)$ is a spacelike vector $\forall s \in I$, then α is called spacelike curve. If α is a spacelike curve and $\left\langle \alpha'(s), \alpha'(s) \right\rangle_L = 1$, then α is called unit speed spacelike curve. Timelike and spacelike curves have non-null Serret Frenet frames [12].

At the same time we know that $\left\langle \alpha'(s), \alpha'(s) \right\rangle_L = 0$ and $\left\langle \alpha''(s), \alpha''(s) \right\rangle_L > 0\ \forall s \in I$, then α is called null curve. If α is a null curve and $\langle \alpha''(s), \alpha''(s) \rangle_L = 1$, then α is called a null curve given by the pseudo arc length parameter. If $\left\langle \alpha'(s), \alpha'(s) \right\rangle_L > 0$ and $\langle \alpha''(s), \alpha''(s) \rangle_L = 0$, $\forall s \in I$, then α is called pseudo-null curve. If α is a pseudo-null curve and $\left\langle \alpha'(s), \alpha'(s) \right\rangle_L = 1$, then α is called a pseudo-null curve given by pseudo arc length parameter [12].

2.1. **Serret Frenet frames of spacelike curves given by arclength parameter.** Let $\alpha : I \to \mathbb{E}_1^3$ be a unit speed spacelike curve. And

$$T(s) = \alpha'(s)$$

is the unit tangent vector of α. Since $T(s)$ is spacelike, $T'(s)$ can be spacelike or timelike. For this reason we will discuss the spacelike curves in two cases. If $T'(s) = \alpha''(s)$ is spacelike, then we have

$$\begin{aligned} \kappa(s) &= \|T'(s)\| = \sqrt{\langle \alpha''(s), \alpha''(s) \rangle_L} \\ N(s) &= \frac{T'(s)}{\kappa(s)}, \\ B(s) &= T(s) \times_L N(s). \end{aligned}$$

Moreover torsion function of the curve α is defined as

$$\tau(s) = -\left\langle N'(s), B(s) \right\rangle_L.$$

Theorem 2.2. *The Frenet formulas for the unit speed spacelike curve $\alpha : I \to \mathbb{E}_1^3$ with spacelike principal normal are as follows*

$$\begin{bmatrix} T'(s) \\ N'(s) \\ B'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}$$

where T, N, B are Frenet vector fields of α and N is spacelike.

If $T'(s) = \alpha^{''}(s)$ is timelike, then we get

$$\begin{aligned}
\kappa(s) &= \|T'(s)\| = \sqrt{-\langle \alpha''(s), \alpha''(s)\rangle_L}, \\
N(s) &= \frac{T'(s)}{\kappa(s)}, \\
B(s) &= T(s) \times_L N(s), \\
\tau(s) &= \left\langle N^{'}(s), B(s)\right\rangle_L.
\end{aligned}$$

Theorem 2.3. *The Frenet formulas for the unit speed spacelike $\alpha : I \to E_1^3$ with timelike principal normal are as follows*

$$\begin{bmatrix} T^{'}(s) \\ N^{'}(s) \\ B^{'}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}$$

where T,N, B are Frenet vector fields of α [16, 17].

2.4. **Serret Frenet frames of pseudo null curves given by pseudo arclength parameter.** Let $\alpha : I \to E_1^3$ be a pseudo-null curve given by pseudo arc length parameter. We know that

$$T(s) = \alpha^{'}(s)$$

is the tangent vector of α and $N(s) = \alpha^{''}(s)$ is a null vector. The binormal vector field B is the unique null vector field orthogonal to T such that

$$\langle N(s), B(s)\rangle_L = 1.$$

If α is straight line then $\kappa(s) = 0$ and in other cases $\kappa(s) = 1$. Furthermore

$$\tau(s) = \left\langle N^{'}(s), B(s)\right\rangle_L.$$

Theorem 2.5. *The Frenet formulas for pseudo-null curve $\alpha : I \to E_1^3$ given by pseudo arc length parameter are as follows*

$$\begin{bmatrix} T^{'}(s) \\ N^{'}(s) \\ B^{'}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ 0 & \tau(s) & 0 \\ -\kappa(s) & 0 & -\tau(s) \end{bmatrix} \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}$$

where T,N, B are Frenet vector fields of α satisfying following relations:

$$\begin{aligned}
\langle N(s), N(s)\rangle_L &= \langle B(s), B(s)\rangle_L = 0, \\
\langle T(s), N(s)\rangle_L &= \langle T(s), B(s)\rangle_L = 0, \\
\langle T(s), T(s)\rangle_L &= \langle N(s), B(s)\rangle_L = 1
\end{aligned}$$

[15, 16].

3. Spacelike W Curve with Spacelike Normal Vector

In this section, we give characterization of spacelike W curve with spacelike normal vector. If $\gamma(s)$ is a spacelike W curve with spacelike normal vector. Then spacelike curve $\gamma(s)$ is given by

$$\gamma(s) = p_0(s)\,T(s) + p_1(s)\,N(s) + p_2(s)\,B(s) \tag{1}$$

for some differentiable functions p_0, p_1 and p_2 of $s \in I \subset \mathbb{R}$. Taking the derivative of both sides of Equation 1 with respect to the arc length parameter and using Serret Frenet formulas which are given by Theorem 1, we get

$$\begin{aligned} p_0'(s) &= \kappa p_1(s) + 1 \\ p_1'(s) &= -\tau p_2(s) - \kappa p_0(s), \\ p_2'(s) &= -\tau p_1(s) \end{aligned} \tag{2}$$

with the use of the equality $\gamma'(s) = T(s)$.

We have the following three cases.

Case 3.1. In this case, the condition of being $\tau^2 - \kappa^2 > 0$ will be examined.

Theorem 3.1. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a twisted spacelike curve, then the position vector $\gamma(s)$ is obtained with the following differentiable functions:*

$$\begin{aligned} p_0(s) &= -\tau c_0 + c_2\kappa \sinh(fs) + c_1\kappa \sinh(fs) + \frac{\tau^2}{f^2}s, \\ p_1(s) &= -c_1 f \sinh(fs) + c_2 f \cosh(fs) + \frac{\kappa}{f^2}, \\ p_2(s) &= \kappa c_0 + c_1\tau \cosh(fs) - c_2\tau \sinh(fs) - \frac{\kappa\tau}{f^2}s, \end{aligned}$$

where $\tau^2 - \kappa^2 = f^2 > 0$ and c_i are arbitrary constants for $0 \le i \le 2$.

Proof. We may write the nonhomogeneous linear differential system of equations in 2, as follows:

$$\begin{bmatrix} p_0'(s) \\ p_1'(s) \\ p_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} p_0(s) \\ p_1(s) \\ p_2(s) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The eigenvalues and eigenvectors of the matrix of nonhomogeneous linear system of the above equation are found as follows:

$$\lambda_1 = 0 \Rightarrow V_1 = \begin{bmatrix} -\tau \\ 0 \\ \kappa \end{bmatrix}, \lambda_2 = f \Rightarrow V_2 = \begin{bmatrix} -\kappa \\ -f \\ \tau \end{bmatrix},$$

$$\lambda_3 = -f \Rightarrow V_3 = \begin{bmatrix} -\kappa \\ f \\ \tau \end{bmatrix}.$$

We obtain the homogenous solution as follows:

$$\begin{aligned} X_h(s) &= c_0 \begin{bmatrix} -\tau \\ 0 \\ \kappa \end{bmatrix} + d_1 e^{fs} \begin{bmatrix} -\kappa \\ -f \\ \tau \end{bmatrix} + d_2 e^{-fs} \begin{bmatrix} -\kappa \\ f \\ \tau \end{bmatrix} \\ &= c_0 \begin{bmatrix} -\tau \\ 0 \\ \kappa \end{bmatrix} + d_1(\cosh(fs) + \sinh(fs)) \begin{bmatrix} -\kappa \\ -f \\ \tau \end{bmatrix} \\ &\quad + d_2(\cosh(fs) - \sinh(fs)) \begin{bmatrix} -\kappa \\ f \\ \tau \end{bmatrix} \end{aligned}$$

where c_0, d_1, d_2 are arbitrary constants. Rewriting the constants as follows

$$d_1 + d_2 = c_1,$$

$$d_1 - d_2 = c_2.$$

We obtain the homogenous solution as

$$X_h(s) = c_0 \begin{bmatrix} -\tau \\ 0 \\ \kappa \end{bmatrix} + c_1 \begin{bmatrix} -\kappa\cosh(fs) \\ -f\sinh(fs) \\ \tau\cosh(fs) \end{bmatrix} + c_2 \begin{bmatrix} -\kappa\sinh(fs) \\ -f\cosh(fs) \\ \tau\sinh(fs) \end{bmatrix}.$$

The fundamental matrix of the nonhomogeneous linear differential system of the equation can be written as

$$\varphi(s) = \begin{bmatrix} -\tau & -\kappa\cosh(fs) & -\kappa\sinh(fs) \\ 0 & -f\sinh(fs) & -f\cosh(fs) \\ \kappa & \tau\cosh(fs) & \tau\sinh(fs) \end{bmatrix}.$$

With the use of the equality $X_p(s) = \varphi(s)\, u(s)$, we may find the vector values function $u(s)$ by following equation

$$\varphi(s)\, u'(s) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Actually, solving the 3×3 linear equation by Crammer's method, we find the particular solution of the Equation 1 as follows:

$$X_p(s) = \begin{bmatrix} \frac{s\tau^2}{f^2} \\ \frac{\kappa}{f^2} \\ -\frac{s\kappa\tau}{f^2} \end{bmatrix}.$$

Since $X_g(s) = X_h(s) + X_p(s)$, then we get the general solution of the system of linear differential equation as follows

$$\begin{aligned} p_0(s) &= -\tau c_0 + c_2\kappa\sinh(fs) + c_1\kappa\sinh(fs) + \frac{\tau^2}{f^2}s, \\ p_1(s) &= -c_1 f\sinh(fs) + c_2 f\cosh(fs) + \frac{\kappa}{f^2}, \\ p_2(s) &= \kappa c_0 + c_1\tau\cosh(fs) - c_2\tau\sinh(fs) - \frac{\kappa\tau}{f^2}s. \end{aligned}$$

□

It is suggested to the readers to see [8] for details of the methods of solving first order nonhomogeneous linear differential systems of equations.

Case 3.2 In this case we take $\tau^2 - \kappa^2 < 0$, we obtain the following theorems in this condition.

Theorem 3.2. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a twisted spacelike curve, then the position vector $\gamma(s)$ is obtained with the following differentiable functions:*

$$\begin{aligned} p_0(s) &= -c_0\tau - c_1\kappa\cos(gs) + c_2\kappa\sin(gs) - \frac{\tau^2}{g^2}s, \\ p_1(s) &= c_1 g\sin(gs) + c_2 g\cos(gs) - \frac{\kappa}{g^2}, \\ p_2(s) &= c_0\kappa + c_1\tau\cos(gs) - c_2\tau\sin(gs) + \frac{\kappa\tau}{g^2}s, \end{aligned}$$

where $\tau^2 - \kappa^2 = -g^2 < 0$ and c_i are arbitrary constants for $0 \le i \le 2$.

Proof. Similar to the proof of Theorem 4, we find the homogeneous solution of the nonhomogeneous linear differential system of equations in 2 as follows:

$$X_h(s) = c_0\begin{bmatrix} -\tau \\ 0 \\ \kappa \end{bmatrix} + c_1\begin{bmatrix} -\kappa\cos(gs) \\ g\sin(gs) \\ \tau\cos(gs) \end{bmatrix} + c_2\begin{bmatrix} \kappa\sin(gs) \\ g\cos(gs) \\ -\tau\sin(gs) \end{bmatrix}$$

Then, we obtain the fundamental matrix as

$$\varphi(s) = \begin{bmatrix} -\tau & -\kappa\cos(gs) & \kappa\sin(gs) \\ 0 & g\sin(gs) & g\cos(gs) \\ \kappa & \tau\cos(gs) & -\tau\sin(gs) \end{bmatrix}.$$

Finally, particular solution of the system 2 can be given by

$$X_p(s) = \begin{bmatrix} -\frac{\tau^2}{g^2}s \\ -\frac{\kappa}{g^2} \\ \frac{\kappa\tau}{g^2}s \end{bmatrix}.$$

Therefore, we obtain

$$\begin{aligned}
p_0(s) &= -c_0\tau - c_1\kappa\cos(gs) + c_2\kappa\sin(gs) - \frac{\tau^2}{g^2}s,\\
p_1(s) &= c_1 g\sin(gs) + c_2 g\cos(gs) - \frac{\kappa}{g^2},\\
p_2(s) &= c_0\kappa + c_1\tau\cos(gs) - c_2\tau\sin(gs) + \frac{\kappa\tau}{g^2}s.
\end{aligned}$$

□

Case 3.3. In this case we take $\tau^2 - \kappa^2 = 0$, and we get the following subcases:
Case 3.3.1. In this subcase, we will investigate condition of being $\kappa = \tau$.

Theorem 3.3. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a twisted curve, then the position vector $\gamma(s)$ is obtained with the following differentiable functions:*

$$\begin{aligned}
p_0(s) &= -c_0\left(\frac{\kappa^2}{2}s^2 - 1\right) + c_1\kappa s - c_2\frac{\kappa^2}{2}s^2 - \frac{\kappa^2}{6}s^3 + s,\\
p_1(s) &= -c_0\kappa s + c_1 - c_2\kappa s - \frac{\kappa}{2}s^2,\\
p_2(s) &= \frac{1}{2}c_0\kappa^2 s^2 - c_1\kappa s + c_2\left(\frac{\kappa^2}{2}s^2 + 1\right) + \frac{\kappa^2}{6}s^3,
\end{aligned}$$

where c_i are arbitrary constants for $0 \le i \le 2$.

Proof. Substituting $\kappa = \tau$ into the Equation 2, we obtain the fundamental matrix as follows:

$$\varphi(s) = \begin{bmatrix} 1 - \frac{1}{2}s^2\kappa^2 & s\kappa & -\frac{1}{2}s^2\kappa^2 \\ -s\kappa & 1 & -s\kappa \\ \frac{1}{2}s^2\kappa^2 & -s\kappa & \frac{1}{2}s^2\kappa^2 + 1 \end{bmatrix}.$$

and so we find the homogeneous solution of this differential equation as follows

$$X_h(s) = c_0\begin{bmatrix} 1 - \frac{1}{2}s^2\kappa^2 \\ -s\kappa \\ \frac{1}{2}s^2\kappa^2 \end{bmatrix} + c_1\begin{bmatrix} s\kappa \\ 1 \\ -s\kappa \end{bmatrix} + c_2\begin{bmatrix} -\frac{1}{2}s^2\kappa^2 \\ -s\kappa \\ \frac{1}{2}s^2\kappa^2 + 1 \end{bmatrix}$$

Then, we find the particular solution of the differential equation as follows

$$X_p(s) = \begin{bmatrix} -\frac{\kappa^2}{6}s^3 + s \\ -\frac{\kappa}{2}s^2 \\ \frac{\kappa^2}{6}s^3 \end{bmatrix}$$

after the calculations, we obtained general solution of the differential equation by

$$\begin{aligned}
p_0(s) &= -c_0\left(\frac{\kappa^2}{2}s^2 - 1\right) + c_1\kappa s - c_2\frac{\kappa^2}{2}s^2 - \frac{\kappa^2}{6}s^3 + s,\\
p_1(s) &= -c_0\kappa s + c_1 - c_2\kappa s - \frac{\kappa}{2}s^2,\\
p_2(s) &= \frac{1}{2}c_0\kappa^2 s^2 - c_1\kappa s + c_2\left(\frac{\kappa^2}{2}s^2 + 1\right) + \frac{\kappa^2}{6}s^3.
\end{aligned}$$

□

Case 3.3.2. In this subcase, we will investigate condition of being $\kappa = -\tau$.

Theorem 3.4. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a twisted curve, then the position vector $\gamma(s)$ is obtained with the following equality:*

$$\begin{aligned} p_0(s) &= -c_0\left(\frac{\kappa^2}{2}s^2 - 1\right) + c_1\kappa s + c_2\frac{\kappa^2}{2}s^2 - \frac{\kappa^2}{6}s^3 + s, \\ p_1(s) &= -c_0\kappa s + c_1 + c_2\kappa s - \frac{\kappa}{2}s^2, \\ p_2(s) &= -c_0\frac{\kappa^2}{2}s^2 + c_1\kappa s + c_2\left(\frac{1}{2}s^2\kappa^2 + 1\right) - \frac{\kappa^2}{6}s^3, \end{aligned}$$

where c_i are arbitrary constants for $0 \le i \le 2$.

Proof. Proof can be done similar to proof of the previous theorem by substituting $\tau = -\kappa$ in the Equation 2. □

Consider the unit speed spacelike W curve $\gamma : I \to \mathbb{E}_1^3$ with the parametrization

$$\gamma(s) = (\frac{1}{2}\sinh s, \frac{1}{2}\cosh s, \frac{\sqrt{5}}{2}s).$$

We obtain the Frenet frame fields as follows:

$$\begin{aligned} T(s) &= (\frac{1}{2}\cosh s, \frac{1}{2}\sinh s, \frac{\sqrt{5}}{2}), \\ N(s) &= (\sinh s, \cosh s, 0), \\ B(s) &= (\frac{\sqrt{5}}{2}\cosh s, \frac{\sqrt{5}}{2}\sinh s, \frac{1}{2}) \end{aligned}$$

where the curvature and torsion of the curve γ are

$$\kappa(s) = \frac{1}{2} \text{ and } \tau(s) = \frac{\sqrt{5}}{2},$$

respectively. Since $\tau^2 - \kappa^2 = 1 > 0$, then we get

$$p_0(s) = \frac{5}{4}s, \;\; p_1(s) = \frac{1}{2}, \;\; p_2(s) = -\frac{\sqrt{5}}{4}s$$

with the use of Theorem 4 where $c_0 = c_1 = c_2 = 0$. It can be easily seen that

$$\gamma(s) = p_0(s)T(s) + p_1(s)N(s) + p_2(s)B(s).$$

4. Spacelike W Curve with Timelike Normal Vector

In this section, we give characterization of spacelike W curve with timelike normal vector. If $\gamma(s)$ is a spacelike W curve with spacelike normal vector. Then spacelike curve $\gamma(s)$ is given by

$$\gamma\left(s\right) = q_0\left(s\right)T\left(s\right) + q_1\left(s\right)N\left(s\right) + q_2\left(s\right)B\left(s\right) \tag{3}$$

for some differentiable functions q_0, q_1 and q_2 of $s \in I \subset \mathbb{R}$. Taking the derivative of both sides of Equation 1 with respect to the arc length parameter and using Serret Frenet Formulas which are given by Theorem 2, we get

$$\begin{aligned} q_0'(s) &= -\kappa q_1(s) + 1, \\ q_1'(s) &= -\kappa q_0(s) - \tau q_2(s), \\ q_2'(s) &= -\tau q_1(s). \end{aligned} \tag{4}$$

Theorem 4.1. *Let $\gamma : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a twisted spacelike curve, then the position vector $\gamma(s)$ is obtained with the following differentiable functions:*

$$\begin{aligned} q_0(s) &= -c_0\tau + c_1\kappa\cosh(ts) + c_2\kappa\sinh(ts) + \frac{\tau^2}{t^2}s, \\ q_1(s) &= -c_1 t\sinh(ts) - c_2 t\cosh(ts) + \frac{\kappa}{t^2}, \\ q_2(s) &= \kappa c_0 + c_1\tau\cosh(ts) + c_2\tau\sinh(ts) - \frac{\kappa\tau}{t^2}s, \end{aligned}$$

where $\tau^2 + \kappa^2 = t^2 > 0$ and c_i are arbitrary constants for $0 \le i \le 2$.

Proof. We may write the nonhomogeneous linear differential system of equations in 4, as follows:

$$\begin{bmatrix} q_0'(s) \\ q_1'(s) \\ q_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & -\kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} q_0(s) \\ q_1(s) \\ q_2(s) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

With the use of eigenvalues and eigenvectors of the coefficient matrix of the above nonhomogeneous linear system of differential equations, we obtain the homogeneous solution as

$$\begin{aligned} X_h(s) = c_0 \begin{bmatrix} -\tau \\ 0 \\ \kappa \end{bmatrix} &+ d_1(\cosh(ts) + \sinh(ts)) \begin{bmatrix} \kappa \\ -t \\ \tau \end{bmatrix} \\ &+ d_2(\cosh(ts) - \sinh(ts)) \begin{bmatrix} \kappa \\ t \\ \tau \end{bmatrix}. \end{aligned}$$

where c_0, $d_1$and d_2 are arbitrary constants. Rewriting the constants as follows

$$\begin{aligned} d_1 + d_2 &= c_1, \\ d_1 - d_2 &= c_2. \end{aligned}$$

We obtain the homogenous solution as

$$X_h(s) = c_0 \begin{bmatrix} -\tau \\ 0 \\ \kappa \end{bmatrix} + c_1 \begin{bmatrix} \kappa\cosh(ts) \\ -t\sinh(ts) \\ \tau\cosh(ts) \end{bmatrix} + c_2 \begin{bmatrix} \kappa\sinh(ts) \\ -t\cosh(ts) \\ \tau\sinh(ts) \end{bmatrix}$$

and the particular solution of the system 4 as

$$X_p(s) = \begin{bmatrix} \frac{\tau^2}{t^2}s \\ \frac{\kappa}{t^2} \\ -\frac{\kappa\tau}{t^2}s \end{bmatrix}.$$

Finally, we get

$$q_0(s) = -c_0\tau + c_1\kappa\cosh(ts) + c_2\kappa\sinh(ts) + \frac{\tau^2}{t^2}s,$$
$$q_1(s) = -c_1 t\sinh(ts) - c_2 t\cosh(ts) + \frac{\kappa}{t^2},$$
$$q_2(s) = \kappa c_0 + c_1\tau\cosh(ts) + c_2\tau\sinh(ts) - \frac{\kappa\tau}{t^2}s.$$

□

Consider the unit speed spacelike W curve $\gamma : I \to \mathbb{E}_1^3$ with the parametrization

$$\gamma(s) = (\cosh s, \frac{\sqrt{2}}{\sqrt{3}}\sinh s, \frac{1}{\sqrt{3}}\sinh s).$$

We obtain the Serret Frenet frame fields as follows:

$$T(s) = (\sinh s, \frac{\sqrt{2}}{\sqrt{3}}\cosh s, \frac{1}{\sqrt{3}}\cosh s),$$
$$N(s) = (\cosh s, \frac{\sqrt{2}}{\sqrt{3}}\sinh s, \frac{1}{\sqrt{3}}\sinh s),$$
$$B(s) = (0, \frac{1}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}})$$

where the curvature and torsion of the curve γ are

$$\kappa(s) = 1 \text{ and } \tau(s) = 0,$$

respectively. Since $\tau^2 + \kappa^2 = 1$, then we get

$$q_0(s) = 0,\ q_1(s) = 1,\ q_2(s) = 0$$

with the use of Theorem 8 where $c_0 = c_1 = c_2 = 0$. It is clear that

$$\gamma(s) = q_0(s)T(s) + q_1(s)N(s) + q_2(s)B(s).$$

5. Pseudo Null W Curve

In this section, we give characterization of a pseudo null W curve $\gamma : I \to \mathbb{E}_1^3$ given by pseudo arc length parameter. The position vector of spacelike curve $\gamma(s)$ is given by

$$\gamma(s) = m_0(s)\,T(s) + m_1(s)\,N(s) + m_2(s)\,B(s) \tag{5}$$

for some differentiable functions m_0, m_1 and m_2 of $s \in I \subset \mathbb{R}$. Let γ be not a straight line. Then the curvature of γ should be equal to "1". Taking the derivative of both sides of Equation 5 with respect to the pseudo arc length parameter and using the Serret-Frenet formulas which is given in Theorem 3, we get

$$\begin{aligned}\gamma'(s) &= (m_0'(s) - m_2(s))\,T(s)\\ &\quad + (m_1'(s) + m_0(s) + \tau(s)\,m_1(s))\,N(s)\\ &\quad + (m_2'(s) - \tau(s)m_2(s))\,B(s).\end{aligned}$$

It is seen that the following relations are satisfied

$$\begin{aligned} m_0'(s) &= 1 + m_2(s), \\ m_1'(s) &= -m_0(s) - \tau m_1(s), \\ m_2'(s) &= \tau m_2(s). \end{aligned} \tag{6}$$

Theorem 5.1. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a pseudo null twisted curve given by pseudo arc length parameter which is not a straight line. If α is a W curve with $\tau \neq 0$, then the position vector $\alpha(s)$ is stated with the following differentiable functions:*

$$\begin{aligned} &m_0(s) = -c_0\tau + c_1(\cosh(\tau s) + \sinh(\tau s)) + 1, \\ &m_1(s) = c_0 - \frac{c_1}{2\tau}(\cosh(\tau s) + \sin h(\tau s)) + c_2(\cosh(\tau s) - \sinh(\tau s)) - \frac{1}{\tau} + \frac{1}{\tau^2}, \\ &m_2(s) = c_1\tau(\cosh(\tau s) + \sinh(\tau s)). \end{aligned}$$

where c_i are arbitrary constants for $0 \leq i \leq 2$.

Proof. Suppose that α is a W pseudo null curve with $\tau \neq 0$. This means that the coefficients of differential equations given in Equation 6 are also constant. We may write the Equations in 6 as follows:

$$\begin{bmatrix} m_0'(s) \\ m_1'(s) \\ m_2'(s) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & -\tau & 0 \\ 0 & 0 & \tau \end{bmatrix} \begin{bmatrix} m_0(s) \\ m_1(s) \\ m_2(s) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The eigenvalues and eigenvectors of the matrix of nonhomogeneous linear system of the above equation are found as follows:

$$\lambda_1 = 0 \Rightarrow V_1 = \begin{bmatrix} -\tau \\ 1 \\ 0 \end{bmatrix}, \lambda_2 = \tau \Rightarrow V_2 = \begin{bmatrix} 1 \\ -\frac{1}{2\tau} \\ \tau \end{bmatrix}, \lambda_3 = -\tau \Rightarrow V_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The homogeneous solution of this differential equation is found as:

$$\begin{aligned} X_h(s) &= c_0 \begin{bmatrix} -\tau \\ 1 \\ 0 \end{bmatrix} + d_1 e^{\tau s} \begin{bmatrix} 1 \\ -\frac{1}{2\tau} \\ \tau \end{bmatrix} + d_2 e^{-\tau s} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ &= c_0 \begin{bmatrix} -\tau \\ 1 \\ 0 \end{bmatrix} + d_1(\cosh(\tau s) + \sinh(\tau s)) \begin{bmatrix} 1 \\ -\frac{1}{2\tau} \\ \tau \end{bmatrix} \\ &+ d_2(\cosh(\tau s) - \sinh(\tau s)) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{aligned}$$

where c_0, d_1, d_2 are arbitrary constants. We obtain the homogenous solution as

$$X_h(s) = c_0 \begin{bmatrix} -\tau \\ 1 \\ 0 \end{bmatrix} + d_1 \begin{bmatrix} \cosh(\tau s) + \sinh(\tau s) \\ -\frac{1}{2\tau}(\cosh(\tau s) + \sinh(\tau s)) \\ \tau(\cosh(\tau s) + \sinh(\tau s)) \end{bmatrix}$$
$$+ c_2 \begin{bmatrix} 0 \\ \cosh(\tau s) - \sinh(\tau s) \\ 0 \end{bmatrix}.$$

The fundamental matrix of the nonhomogeneous linear differential system of the equation can be written as

$$\varphi(s) = \begin{bmatrix} -\tau & \cosh(\tau s) + \sinh(\tau s) & 0 \\ 1 & -\frac{1}{2\tau}(\cosh(\tau s) + \sinh(\tau s)) & \cosh(\tau s) - \sinh(\tau s) \\ 0 & \tau(\cosh(\tau s) + \sinh(\tau s)) & 0 \end{bmatrix}.$$

With the use of the equality $X_p(s) = \varphi(s)\, u(s)$, we may find the vector values function $u(s)$ by following equation

$$\varphi(s)\, u'(s) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Actually, solving the 3×3 linear equation by Crammer's method, we find the particular solution of the Equation 6 as follows:

$$X_p(s) = \begin{bmatrix} d_2(\cosh(\tau s) + \sinh(\tau s)) + 1 \\ -\frac{d_2}{2\tau}(\cosh(\tau s) + \sinh(\tau s)) - \frac{1}{\tau} + \frac{1}{\tau^2} \\ d_2\tau(\cosh(\tau s) + \sinh(\tau s)) \end{bmatrix}.$$

Since $X_g(s) = X_h(s) + X_p(s)$ and by substituting $d_1 + d_2 = c_1$,then it is seen that

$$\begin{aligned}
m_0(s) &= -c_0\tau + c_1(\cosh(\tau s) + \sinh(\tau s)) + 1, \\
m_1(s) &= c_0 - \frac{c_1}{2\tau}(\cosh(\tau s) + \sin h(\tau s)) + c_2(\cosh(\tau s) - \sinh(\tau s)) - \frac{1}{\tau} + \frac{1}{\tau^2}, \\
m_2(s) &= c_1\tau(\cosh(\tau s) + \sinh(\tau s)).
\end{aligned}$$

$\square$

Theorem 5.2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a pseudo null twisted curve given by pseudo arc length parameter which is not a straight line. If α is a planar W curve , then the position vector $\alpha(s)$ is stated with the following differentiable functions:*

$$\begin{aligned}
m_0(s) &= c_0 s + c_1 + s, \\
m_1(s) &= -c_0\frac{s^2}{2} - c_1 s + c_2 - \frac{s^2}{2}, \\
m_2(s) &= c_0,
\end{aligned}$$

where c_i are arbitrary constants for $0 \le i \le 2$.

Proof. Suppose that α is a planar W pseudo null curve. That is $\tau = 0$. We may write the equations in 6 as follows:

$$m_0'(s) = m_2(s) + 1,$$
$$m_1'(s) = -m_0(s),$$
$$m_2'(s) = 0.$$

Then we can easily find

$$m_0(s) = c_0 s + c_1 + s$$
$$m_1(s) = -c_0 \frac{s^2}{2} - c_1 s + c_2 - \frac{s^2}{2},$$
$$m_2(s) = c_0,$$

where c_i are arbitrary constants for $0 \leq i \leq 2$. $\square$

Consider the pseudo null W curve $\alpha : I \rightarrow \mathbb{E}_1^3$ with the parametrization

$$\alpha(s) = (\frac{s^2}{2}, \frac{s^2}{2}, s).$$

We obtain the Serret Frenet frame fields as follows:

$$T(s) = (s, s, 1),$$
$$N(s) = (1, 1, 0),$$
$$B(s) = (\frac{s^2 - 1}{2}, \frac{1 - s^2}{2}, s)$$

where the curvature and torsion of the curve γ are

$$\kappa(s) = 1 \text{ and } \tau(s) = 0,$$

respectively. Thus we have

$$m_0(s) = s, m_1(s) = -\frac{s^2}{2}, \; m_2(s) = 0$$

with the use of Theorem 10 where $c_0 = c_1 = c_2 = 0$. Actually, it is seen that

$$\alpha(s) = m_0(s)T(s) + m_1(s)N(s) + m_2(s)B(s).$$

6. Conclusion

According to all findings of this paper, we can summarize the following results in the table:

ihF4.1615in2.8115in0inFigure

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