On Boolean, Heyting and Brouwerian Algebras

Mohammad Nawaz

Balochistan University of Information Technology Engineering and Management Sciences, Quetta.

Abstract

Several models for logic have been developed in previous century. Each model has its own algebra of truth values. Frequently used algebras for truth values are Boolean Algebra for classical logic and Heyting Algebra and Brouwerian algebra for intuitionist logic logic and Brazilian logic respectively. Each of these algebras is a distributive lattice. In this paper we consider lattices which admit certain binary operations that force distributivity. For Boolean algebra this binary operation is induced by an endofunction which turns out to be negation for the Boolean algebra. These binary operations and corresponding negations for Heyting algebras and Browerian algebras, are discussed in detail. At the end we give a necessary and sufficient condition for a lattice to be a Boolean algebra.

Keywords: Boolean algebra, Heyting algebra, Brouwerian algebra, negation, excluded middle.

Corresponding author's emial: nawaz@buitms.edu.pk

1. INTRODUCTION

Truth tables for the classical connectives form a valuation system in the sense of [1] with the Boolean algebra {0, 1} as algebra of truth values, where 0 is interpreted as "false" and 1 as "true". In this logic only possible truth values are true and false. On the level of equivalence and equations the subjects of propositional logic and Boolean algebras are essentially the same. Fuzzy logic uses interval [0, 1] for truth values and the Brazilian logic uses the Brouwerian algebra. Presently mathematicians are attempting to develop a model, known as Quantal Logic [2], for non-commutative context. Algebra of truth values for this model of logic is known as Quantale [2, 3].

Law of excluded middle is valid in Classical logic (modeled by Boolean algebra) which is consequence of $a \vee \rceil a = 1$ that holds in any Boolean algebra. Other laws valid in classical logic include ex falso quodlibet, De Morgan's laws, elimination of double negation and introduction of double negation which are consequences of $a \wedge \rceil a = 0$, $\rceil(a \wedge b) = \rceil a \vee \rceil b$, $\rceil(a \vee b) = \rceil a \wedge \rceil b$ and $\rceil\rceil a = a$.

Proofs that appeal to the law of excluded middle mostly do not give a way of deciding that which alternative holds and mathematicians keep on striving for more neat proofs. An example of proof by appeal to excluded middle is the proof [1, 4] for the statement that there is solution of $x^y = z$ with x and y irrational and z rational (call" there is solution of $x^y = z$ with x and y irrational and z rational" as ϕ). The proof goes as follows.

If $\sqrt{2}^{\sqrt{2}}$ is rational (call "$\sqrt{2}^{\sqrt{2}}$ is rational" asψ) then there is solution of $x^y = z$ with $x = \sqrt{2}$, $y = \sqrt{2}$ and $z = \sqrt{2}^{\sqrt{2}}$. If $\sqrt{2}^{\sqrt{2}}$ is irrational (that is $\urcorner\psi$) then there is solution of $x^y = z$ with $x = \sqrt{2}^{\sqrt{2}}$, $y = \sqrt{2}$ and $z = 2$.

Note that we are not given a way to decide that which alternative holds. This is a serious flaw. Both of the references [1] and [4] do not mention a constructive proof for "There is solution of $x^y = z$ with x and y irrational and z rational"**.** One is left with feeling that perhaps constructive proof for this statement is not known. However there is a much simpler and neat solution to this problem. Just take $x = \sqrt{2}$ and $y = \log_2 25$. Then x^y (= 5) is rational.

Heyting algebras are algebras of truth values for valuation systems for Intuitionistic logic. They play a role in intutionistic logic analogous to that of Boolean algebras in classical Logic. In Intuitionistic logic [4] $a \vee \urcorner a = 1$, $\urcorner(a \wedge b) = \urcorner a \vee \urcorner b$ and $\urcorner\urcorner a \le a$ do not hold however $a \wedge \urcorner a = 0$, $\urcorner(a \vee b) = \urcorner a \wedge \urcorner b$ and $a \le \urcorner\urcorner a$ are valid. Hence excluded middle, one of De Morgan's Laws and elimination of double negation are not valid in intuitionistic logic, while intuitionistic negation satisfies ex falso quodlibet, introduction of double negation and one of the De Morgan's Laws.

The statement $\psi \vee \urcorner \psi$ is not valid intuitionistically. This means, in particular, that in intuitionistic logic we are not entitled to infer ϕ from $\psi \to \phi$ and $\urcorner \psi \to \phi$. However, in classical logic, we infer ϕ from $\psi \to \phi$ and $\urcorner \psi \to \phi$ (which is an instance of excluded middle).

Dual to intuitionistic logic is Brazilian Logic which is modeled by Brouwerian algebra. For any a in a Brouwerian algebra, it is the case that $a \vee \urcorner a = 1$, $\urcorner(a \wedge b) = \urcorner a \vee \urcorner b$ and $\urcorner\urcorner a \le a$ while $a \wedge \urcorner a = 0$, $\urcorner(a \vee b) = \urcorner a \wedge \urcorner b$ and $a \le \urcorner\urcorner a$ may fail.

Thus for the Brouwerian negation, law of excluded middle, one of Demorgan's Laws and elimination of double negation are valid while law of ex falso quodlibet, one Demorgan's Law and introduction of double negation are not. Brouwerian algebra is also a good home for Paraconsistent logic as a slight overlap is there in a and $\urcorner a$ which is in conformity with "in the semantics of many paraconsistent logics, truth and falsity may overlap" [5]. It is pertinent to recall here that in his Nobel Lecture [6] Harold Pinter said "There are no hard distinctions between what is real and what is unreal, nor between what is true and what is false. A thing is not necessarily either true or false; it can be both true and false."

It is important to note here that $\urcorner$ has different meaning for each of the cases mentioned above. For a Boolean algebra $\urcorner a$ is just the complement of a, for Heyting algebra of open subsets of a topological space $\urcorner a$ stands for interior of complement of a and for Brouwerian algebra of closed subsets of a topological space $\urcorner a$ is closure of complement of a. Phrases like pseudo-complementation and supplementation are also used for $\urcorner$ respectively in the context of Heyting algebra and Brouwerian algebra. In this paper we shall denote these negations by h(a,0) and g(a,1) where h and g are binary operations on the corresponding Heyting algebra and the Brouwerian algebra respectively.

Some authors use the term Brouwerian algebra to mean the structure we have defined as Heyting algebra [7], however our definition is in conformity with work published in recent years [8, 9, 10].

2. Preliminaries

We recall few definitions and propositions from [11] and [12].

Definition 2.1: A lattice is a set with two binary operations $\vee$, $\wedge$ and two distinguished elements 0, 1 respectively as their unit elements, and that $\vee$ and $\wedge$ are associative, commutative and idempotent and $\vee$ and $\wedge$ satisfy absorptive laws.

Definition 2.2: A **Boolean algebra** is a distributive lattice A which is equipped with an additional unary operation $\rceil : A \longrightarrow A$ such that $\rceil a \wedge a = 0$ and $\rceil a \vee a = 1$.

Definition 2.3: Heyting algebra [12, 13] is a lattice A equipped with a binary operation

$h : A \times A \longrightarrow A$ defined as $c \leq h(a, b)$ iff $c \wedge a \leq b$.

Dual to the concept of Heyting algebra is the concept of Brouwerian algebra.

Definition 2.4: Brouwerian algebra is a lattice A equipped with a binary operation

$g : A \times A \longrightarrow A$ defined as $g(a, b) \leq c \Leftrightarrow b \leq a \vee c$.

For any lattice A the inequalities

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ and $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$

are trivially valid because for the former $b \leq a \vee b$ and $c \leq a \vee c$ gives

$b \wedge c \leq (a \vee b) \wedge (a \vee c)$, also $a \leq (a \vee b) \wedge (a \vee c)$, therefore $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$.

For the later $b, c \leq b \vee c$ gives $a \wedge b \leq a \wedge (b \vee c)$ and $a \wedge c \leq a \wedge (b \vee c)$ therefore

$(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$.

A lattice A is distributive if $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for all $a, b, c \in A$,
We note that, if above law holds then so does its dual i.e. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and vice versa.

Proposition 2.1: If a, b, c are any three elements of a distributive lattice A then there exists at the most one $x \in A$ such that $x \wedge a = b$ and $x \vee a = c$.

Proposition 2.2: For a Boolean algebra A

$$c \leq \rceil a \vee b \Rightarrow a \wedge c \leq b \text{ and } a \wedge c \leq b \Rightarrow c \leq \rceil a \vee b.$$

The first implication is the consequence of $\rceil a \wedge a = 0$ while the second follows from $\rceil a \vee a = 1$. Proof requires distributivity of A.

3. Results

Proposition 3.1: If a lattice A admits an endofunction $f : A \longrightarrow A$ such that

$c \leq f(a) \vee b \Leftrightarrow a \wedge c \leq b$ for all $a, b, c \in A$. Then

i. A is distributive

ii. For all $a, b, c \in A$, $c \leq f(a) \vee b \Rightarrow a \wedge c \leq b$ iff $a \wedge f(a) = 0$

iii. For all $a, b, c \in A$, $a \wedge c \leq b \Rightarrow c \leq f(a) \vee b$ iff $a \vee f(a) = 1$

Proof:

i.

It only remains to establish that $a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$.

Defining property of f combined with $a \wedge b \leq a \wedge b$ and $a \wedge c \leq a \wedge c$ gives

$b \leq f(a) \vee (a \wedge b)$ and $c \leq f(a) \vee (a \wedge c)$.

Therefore $b \vee c \leq f(a) \vee ((a \wedge b) \vee (a \wedge c))$. Hence $a \wedge (b \vee c) \leq (a \wedge b) \vee (a \wedge c)$.

ii.

$c \leq f(a) \vee b \Rightarrow a \wedge c \leq a \wedge (f(a) \vee b) = (a \wedge f(a)) \vee (a \wedge b) = a \wedge b \leq b$. We note that this proof required distributivity.

For converse let $c = f(a)$ and $b = 0$ then $f(a) \leq f(a) \vee 0$.

Therefore $a \wedge f(a) \leq 0$. And hence $a \wedge f(a) = 0$.

iii.

$a \wedge c \leq b \Rightarrow f(a) \vee (a \wedge c) \leq f(a) \vee b \Rightarrow (f(a) \vee a) \wedge (f(a) \vee c) \leq f(a) \vee b$

$\Rightarrow f(a) \vee c \leq f(a) \vee b \Rightarrow c \leq f(a) \vee b$. Proof of this part also depends on the distributive law.

Conversely since $a \wedge 1 \leq a$ therefore $1 \leq f(a) \vee a$ and hence $f(a) \vee a = 1$.

Corollary 3.1.1: For any lattice A the following are equivalent

i. A is distributive and admits an endofunction f: A $\longrightarrow$ A such that $a \wedge f(a) = 0$ and $a \vee f(a) = 1$

ii. A admits an endofunction f : A $\longrightarrow$ A such that $c \leq f(a) \vee b \Leftrightarrow a \wedge c \leq b$.

Proposition 3.2: A lattice A is a Boolean algebra if and only if it admits an endofunction f: A $\longrightarrow$ A such that $c \leq f(a) \vee b \Leftrightarrow a \wedge c \leq b$ for all $a, b, c \in A$

Corollary 3.2.1. In a Boolean algebra A

1. The following are equivalent

   i. $\neg a \wedge a = 0$

   ii. $c \leq \neg a \vee b \Rightarrow a \wedge c \leq b$

2. The following are equivalent

   i. $\neg a \vee a = 1$

   ii. $a \wedge c \leq b \Rightarrow c \leq \neg a \vee b$

Corollary 3.2.2. In a Boolean algebra A

$c \leq \neg a \vee b \Leftrightarrow a \wedge c \leq b$

It is well known that every Boolean algebra A becomes a Heyting algebra with

h : A x A $\longrightarrow$ A defined as $h(a, b) = \neg a \vee b$.

Also for a Heyting algebra A we may define

$\neg$: A $\longrightarrow$ A as $\neg a = h(a, 0)$.

But this does not make A into a Boolean algebra. In fact the condition $(a \wedge c \leq b \Rightarrow c \leq \neg a \vee b)$ of above theorem does not hold in this case. For if it does, then in particular for $c = 1$ and $b = a$ we obtain $1 \leq \neg a \vee a$ (because $a \wedge 1 \leq a$) which is not true in general as the lattice Ω of open subsets of a topological space is a Heyting algebra in which

$\rceil a = h(a, 0)$ stands for the exterior of a and we know that union of an open set with its exterior does not give the whole space in general. For further details see [12] which also discusses a sublattice A of P(X) (the power set of an infinite set X) consisting of all finite subsets of X together with X itself. A is not a Heyting algebra because $h(a, 0)$ is not defined for a non-empty subset a of X. This sublattice A of P(X) is in fact the lattice of closed subsets of co-finite topology on X and is an example of following concept.

Lattice of closed subsets of a topological space is Brouwerian algebra. For closed sets of a topological space $g(a, b) \leq c \Leftrightarrow b \leq a \vee c$ means that g(a , b) is the smallest closed set whose union with a contains b.

In particular we have $g(a, 1) \leq c \Leftrightarrow 1 = a \vee c$. For closed sets this means that $g(a, 1)$ is the smallest closed set whose union with a is 1.

Proposition 3.3: If lattice A admits a binary operation $g: A \times A \longrightarrow A$ such that $g(a, b) \leq c \Leftrightarrow b \leq a \vee b$ for all $a, b, c \in A$. Then A is distributive.

Proof**:** It only remains to establish that $(a \vee b) \wedge (a \vee c) \leq a \vee (b \wedge c)$.

Since $(a \vee b) \wedge (a \vee c) \leq a \vee b$ therefore $g(a, (a \vee b) \wedge (a \vee c)) \leq b$

Similarly $g(a, (a \vee b) \wedge (a \vee c)) \leq c$, we thus conclude that $g(a, (a \vee b) \wedge (a \vee c)) \leq b \wedge c$.

Therefore $(a \vee b) \wedge (a \vee c) \leq a \vee (b \wedge c)$

Proposition 3.4:

Let A be a lattice, g be a binary operation on A, then the following are equivalent

(a) $g(a, b) \leq c$ iff $b \leq a \vee c$

(b) i. $g(a, a) = 0$ ii. $a \vee g(a, b) = a \vee b$

iii. $b \vee g(a, b) = b$ iv. $g(a, b \vee c) = g(a, c) \vee g(a, b)$

Proof: Suppose (a) holds, then

i. $a \leq a \vee 0 \Rightarrow g(a, a) \leq 0$

ii. From definition of g we have $b \leq a \vee g(a, b)$. Therefore $a \vee b \leq a \vee g(a, b)$.

Also $b \leq a \vee b$, therefore $g(a, b) \leq b$ and hence $a \vee g(a, b) \leq a \vee b$.

iii. As above, $b \leq a \vee b$ gives $g(a, b) \leq b$. Therefore $b \vee g(a, b) = b$.

iv. Again by definition of g we have $b \leq a \vee g(a, b)$ and $c \leq a \vee g(a, c)$.

Therefore $b \vee c \leq a \vee g(a, b) \vee g(a, c)$. Hence $g(a, b \vee c) \leq g(a, b) \vee g(a, c)$.

Also $g(a, b \vee c) \leq g(a, b \vee c)$ gives $b \vee c \leq a \vee g(a, b \vee c)$.

Now from $b \leq b \vee c \leq a \vee g(a, b \vee c)$ we have $g(a, b) \leq g(a, b \vee c)$.

Similarly $g(a, c) \leq g(a, b \vee c)$. Hence $g(a, b) \vee g(a, c) \leq g(a, b \vee c)$.

Conversely suppose that (b) holds and $g(a, b) \leq c$. Then from (ii) $b \leq a \vee g(a, b)$.

Therefore $b \leq a \vee g(a, b) \leq a \vee c$. Therefore $b \leq a \vee c$.

Now suppose that $b \leq a \vee c$. Then from (iii)

$c = c \vee g(a , c) \geq g(a , a) \vee g(a , c) = g(a , a \vee c) \geq g(a , b)$

Because g(a , -) is order preserving (this follows from iv). Hence $g(a , b) \leq c$ iff $b \leq a \vee c$

Corollary 3.4.1: $a \vee g(a , 1) = 1$

Proof: Since $g(a , 1) \leq g(a , 1)$ therefore $1 \leq a \vee g(a , 1)$ and therefore $a \vee g(a , 1) = 1$.

(This means $a \vee \rceil a = 1$ where $\rceil a$ is closure of complement of a)

Note: $a \wedge g(a , 1) = 0$ is not true in general because for a, closed in a topology, $a \wedge g(a , 1)$ may not be empty.

Proposition 3.5:

In a Boolean algebra A

1. TFAE

   i. $g(a, b) \leq c$ iff $b \leq a \vee c$ ii. $g(a, b) = \rceil a \wedge b$

2. TFAE

   i. $c \leq h(a, b)$ iff $a \wedge c \leq b$ ii. $h(a, b) = \rceil a \vee b$

Proof: 1. i. $\Rightarrow$ ii.

Since A is Boolean algebra, we have $\rceil a \vee a = 1$ and $\rceil a \wedge a = 0$.

We shall prove that $g(a, b) = \rceil a \wedge b$.

Now $a \vee (\rceil a \wedge b) = (a \vee \rceil a) \wedge (a \vee b) = 1 \wedge (a \vee b) = a \vee b$ gives $b \leq a \vee (\rceil a \wedge b)$.

Thus $g(a, b) \leq \rceil a \wedge b$.

Also $g(a, b) \leq g(a, b)$ gives $b \leq a \vee g(a, b)$

Therefore $\rceil a \wedge b \leq \rceil a \wedge (a \vee g(a, b)) = (\rceil a \wedge a) \vee (\rceil a \wedge g(a, b)) = 0 \vee (\rceil a \wedge g(a, b))$

$= \rceil a \wedge g(a, b) \leq g(a, b)$.

Hence $g(a, b) = \rceil a \wedge b$.

ii. $\Rightarrow$ i. Suppose $\rceil a \wedge b \leq c$ then $a \vee (\rceil a \wedge b) \leq a \vee c$ therefore $(a \vee \rceil a) \wedge (a \vee b) \leq a \vee c$ and hence $1 \wedge (a \vee b) \leq a \vee c$ so that $(a \vee b) \leq a \vee c$ therefore $b \leq a \vee b \leq a \vee c$.

Now suppose that $b \leq a \vee c$ then $\rceil a \wedge b \leq \rceil a \wedge (a \vee c)$. Applying distributive law and $\rceil a \wedge a = 0$ we get $\rceil a \wedge b \leq \rceil a \wedge c \leq c$ and hence $\rceil a \wedge b \leq c$.

2. i. $\Rightarrow$ ii. We shall prove that $h(a, b) = \rceil a \vee b$

Since $a \wedge (\rceil a \vee b) = (a \wedge \rceil a) \vee (a \wedge b) = (a \wedge b) \leq b$ therefore $\rceil a \vee b \leq h(a, b)$.

Also $h(a, b) \leq h(a, b)$ gives $a \wedge h(a, b) \leq b$.

Therefore $\rceil a \vee (a \wedge h(a, b)) \leq \rceil a \vee b$ and using distributive law and $\rceil a \vee a = 1$ on the left hand side we obtain $\rceil a \vee h(a, b) \leq \rceil a \vee b$ which gives $h(a, b) \leq \rceil a \vee h(a, b) \leq \rceil a \vee b$.

Hence $h(a, b) = \rceil a \vee b$.

ii.$\Rightarrow$ i. Suppose that $c \leq \rceil a \vee b$ then $a \wedge c \leq a \wedge (\rceil a \vee b)$. Using distributive law and $\rceil a \wedge a = 0$ on the right hand side we obtain $a \wedge c \leq a \wedge b \leq b$, hence $a \wedge c \leq b$.

Now suppose that $a \wedge c \leq b$ then $\rceil a \vee (a \wedge c) \leq \rceil a \vee b$. Again the distributive law and $\rceil a \vee a = 1$ give $c \leq \rceil a \vee c \leq \rceil a \vee b$ and hence $c \leq \rceil a \vee b$.

Corollary 3.5.1: Every Boolean algebra is a Brouwerian algebra as well as Heyting algebra.

Corollary 3.5.2: For a Boolean algebra A

$$\mathrm{g}(a, 1) = \mathrm{h}(a, 0) = \rceil a.$$

Proposition 3.6: If a lattice A admits binary operations g and h as described above then $\mathrm{h}(a, 0) \leq \mathrm{g}(a, 1)$.

Proof: From the definitions of g and h, we have

$$\mathrm{g}(a, 1) \leq c \text{ iff } 1 \leq a \vee c \text{ .... } (1)$$

$$c \leq \mathrm{h}(a, 0) \text{ iff } a \wedge c \leq 0 \text{ ....}(2)$$

These two in particular give $a \wedge \mathrm{h}(a, 0) = 0$ and $a \vee \mathrm{g}(a, 1) = 1$.

Now $a \wedge \mathrm{h}(a, 0) = 0$ gives $(a \wedge \mathrm{h}(a, 0)) \vee \mathrm{g}(a, 1) = 0 \vee \mathrm{g}(a, 1)$.

Applying distributive law we obtain $(a \vee \mathrm{g}(a, 1)) \wedge (\mathrm{h}(a, 0) \vee \mathrm{g}(a, 1)) = \mathrm{g}(a, 1)$.

And using $a \vee \mathrm{g}(a, 1) = 1$ we get $\mathrm{h}(a, 0) \vee \mathrm{g}(a, 1) = \mathrm{g}(a, 1)$ hence $\mathrm{h}(a, 0) \leq \mathrm{g}(a, 1)$.

Proposition 3.7: TFAE

i. $a \vee \mathrm{h}(a, 0) = 1$

ii. $a \wedge \mathrm{g}(a, 1) = 0$

iii. $\mathrm{g}(a, 1) = \mathrm{h}(a, 0)$.

Proof: i. $\Rightarrow$ ii. Suppose $a \vee \mathrm{h}(a, 0) = 1$ then $1 \leq a \vee \mathrm{h}(a, 0)$ and therefore $\mathrm{g}(a, 1) \leq \mathrm{h}(a, 0)$ by (1) above. This gives $a \wedge \mathrm{g}(a, 1) \leq 0$ by (2). Therefore $a \wedge \mathrm{g}(a, 1) = 0$.

ii. $\Rightarrow$ iii. Suppose $a \wedge \mathrm{g}(a, 1) = 0$ then $a \wedge \mathrm{g}(a, 1) \leq 0$. Then (2) gives $\mathrm{g}(a, 1) \leq \mathrm{h}(a, 0)$ and hence $\mathrm{g}(a, 1) = \mathrm{h}(\mathbf{a}, 0)$ by proposition 3.6.

iii. $\Rightarrow$ i. Suppose $\mathrm{h}(a, 0) = \mathrm{g}(a, 1)$ then $a \vee \mathrm{h}(a, 0) = a \vee \mathrm{g}(a, 1)$ and therefore $a \vee \mathrm{h}(a, 0) = 1$.

Proposition 3.8: A lattice A equipped with binary operations g and h as above is a Boolean algebra if and only if any one of above equivalent conditions is satisfied.

Proof: Simple, define $\rceil a = \mathrm{g}(a, 1)$ (or equivalently $\rceil a = \mathrm{h}(a, 0)$).

We note that $\mathrm{g}(a \vee b, 1) = \mathrm{g}(a, 1) \wedge \mathrm{g}(b, 1)$ may not hold in a Brouwerian algebra, however, we have

Proposition 3.9: $\mathrm{g}(a \wedge b, 1) = \mathrm{g}(a, 1) \vee \mathrm{g}(b, 1)$

Proof: $(a \wedge b) \vee \mathrm{g}(a, 1) = (a \vee \mathrm{g}(a, 1)) \wedge (b \vee \mathrm{g}(a, 1)) = 1 \wedge (b \vee \mathrm{g}(a, 1)) = b \vee \mathrm{g}(a, 1)$. Similarly $(a \wedge b) \vee \mathrm{g}(b, 1) = a \vee \mathrm{g}(b, 1)$. Therefore $(a \wedge b) \vee \mathrm{g}(a, 1) \vee \mathrm{g}(b, 1) = b \vee \mathrm{g}(a, 1) \vee a \vee \mathrm{g}(b, 1) = a \vee \mathrm{g}(a, 1) \vee b \vee \mathrm{g}(b, 1) = 1$ (by corollary 3.4.1). And therefore $1 \leq (a \wedge b) \vee \mathrm{g}(a, 1) \vee \mathrm{g}(b, 1)$ which gives

$$\mathrm{g}(a \wedge b, 1) \leq \mathrm{g}(a, 1) \vee \mathrm{g}(b, 1) \text{ .... } (3)$$

Also $(a \wedge b) \vee g(a \wedge b, 1) = 1$ gives $(a \vee g(a \wedge b, 1)) \wedge (b \vee g(a \wedge b, 1)) = 1$. And therefore $a \vee g(a \wedge b, 1) = 1$ and $b \vee g(a \wedge b, 1) = 1$. Or $1 \leq a \vee g(a \wedge b, 1)$ and $1 \leq b \vee g(a \wedge b, 1)$.

Therefore $g(a, 1) \leq g(a \wedge b, 1)$ and $g(b, 1) \leq g(a \wedge b, 1)$. This gives

$g(a, 1) \vee g(b, 1) \leq g(a \wedge b, 1)$ ……………….... (4)

From (3) and (4) we obtain $g(a \wedge b, 1) = g(a, 1) \vee g(b, 1)$.

References:

[1] Dummett M. (1977). Oxford Logic Guides *Elements of Intuitionism* Oxford University Press.

[2] Mulvey CJ and Nawaz M. (1995). *Quantales: Quantal Sets*, Non- Classical Logics and Their Applications to Fuzzy Subsets. 159-217, Kluwer Academic Publishers, Netherlands.

[3] Nawaz M. (2007). *Quasi-symmetric B-categories,* Demonstratio Mathematica, 40(3): 517-527.

[4] Troelstra AS. *Aspects of Constructive Mathematics,* Handbook of Mathematical Logic, North Holland Publishing Company 1977.

[5] Priest G. "Dialetheism", *The Stanford Encyclopedia of Philosophy (Summer 2004 Edition)*, Edward N. Zalta (ed.).

[6] Harold P. *Art, Truth & Politics,* Nobelprize.org the Official Web Site of the Nobel Prize, 2005.

[7] Hecht T and Katrinak T. *Equational Classes of Relative Stone Algebras*, Notre Dame Journal of Formal Logic, 8(2), 1972.

[8] Miller D. (1991) *An Open Problem in Tarski's Calculus of Deductive Systems*, Bulletin of the Section of Logic, 20(2): 36-43.

[9] Rachunek J, Olomouc, *A duality between Algebras of Basic Logic and Bounded Representable DRl-Monoids*, Mathematica Bohemica 3. 561-569.

[10] Hartmann S, Link S.(2004). *Multi-valued Dependencies in the presence of Lists*, ACM SIGMOD/PODS Conference, Maison de la Chimie, Paris, France, June 13-18.

[11] Birkhoff G. (1967). *Lattice Theory*, A.M.S. Colloquium Publications vol. Xxv, American Mathematical Society, Providence, (third edition).

[12] Johnstone PT. *Stone Spaces*, Cambridge studies in advanced Mathematics 3, Cambridge University Press 1977.

[13] Fourman MP, Scott DS, *Sheaves and Logic*, Applications of Sheaves, lecture notes in Mathematics 753, Springer Verlag 1979.