Goldbach Primes Associated With 2n

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Abstract

For 2n = p + q where p and q are primes, the pair (p , q) is called Goldbach pair (or Goldbach partition) and any constituent prime of a Goldbach pair for 2n will be called the Goldbach prime associated with 2n. The Goldbach primes associated with 2n are distributed evenly on both sides of n. In this paper we show that the number of Goldbach primes associated with 2n is odd if and only if n is prime. We also prove that if p is a Goldbach prime associated with 2n then any prime q is Goldbach prime associated with 2n + (q – p).

Keywords: prime, Goldbach pair, ceiling function.

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Introduction

Goldbach conjecture [4, 7] is the oldest unsolved problem of Mathematics after settlement of Fermat's Last Theorem in 1994 by Andrew Wiles [6]. It states that every even number ≥ 4 can be written as sum of two primes (or every even number greater than 4 can be written as sum of two odd primes). With every even number 2n we assign a set, the set of Goldbach primes associated with 2n, given by
$B(2n) = \{p \mid p \text{ and } 2n - p \text{ are primes}\}$

The Goldbach conjecture may now be stated as B (2n) is not empty for $n \geq 2$. It is obvious that B (2n) is finite and its members can be written in ascending (descending) order. Occurrence of larger primes as least members of B (2n) is rare. For example, for $n < 10^{10}$, B (2n) does not contain a prime larger than 2017 as its least member [3].

We have the following proposition
Proposition 1: For any $p \in B(2n)$ there exists r such that $p = n - r$ or $p = n + r$.
Proof: Take $r = |n - p|$.

Proposition 2: $n - r \in B(2n)$ iff $n + r \in B(2n)$
Proof:
$$\begin{aligned} n - r \in B(2n) &\Leftrightarrow n - r \text{ and } 2n - (n - r) \text{ are primes} \\ &\Leftrightarrow n - r \text{ and } n + r \text{ are primes} \\ &\Leftrightarrow n + r \text{ and } n - r \text{ are primes} \\ &\Leftrightarrow n + r \text{ and } 2n - (n + r) \text{ are primes} \\ &\Leftrightarrow n + r \in B(2n) \end{aligned}$$ □

A Goldbach pair associated with an even number 2n is an ordered pair (p, q), $p \leq q$, both p and q primes, and p + q = 2n. If (p, q) is Goldbach pair associated with 2n then [3, 5] call p + q as Goldbach partition of 2n.

Corollary: Each Goldbach pair for 2n can be written as $(n - r, n + r)$ where $0 \leq r < n$. □

This proposition shows that the Goldbach primes associated with 2n are evenly distributed on both sides of n. For example

B (36) ={5, 7, 13, 17, 19, 23, 29, 31}

5____7____13____17____ ↓ ____19____23____29____31
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And

B (52) = {5, 11, 23, 29, 41, 47}

5____11____23____ ↓ ____29____41____47
26

The Goldbach pairs for 36 are (5, 31), (7, 29), (13, 23), (17, 19) and for 52 are (5, 47), (11, 41), (23, 29). All the primes between n and 2n exist in B(2n) for n = 210 and the number of Goldbach pairs is less than (0.961) $\left(\frac{n}{\log n}\right)$ for $n \geq 10^{24}$ [2].

The fact that first and second primes of a Goldbach pair for 2n are at equal distance from n is also reflected in the table of addition modulo 2n on B(2n) in the form of symmetry about main diagonal. These tables are given below for 36 and 52.

Table 1 Addition modulo 36 on B (36)

$\oplus_{36}$	5	7	13	17	19	23	29	31
5	10	12	18	22	24	28	34	0
7	12	14	20	24	26	30	0	2
13	18	20	26	30	32	0	6	8
17	22	24	30	34	0	4	10	12
19	24	26	32	0	2	6	12	14
23	28	30	0	4	6	10	16	18
29	34	0	6	10	12	16	22	24
31	0	2	8	12	14	18	24	26

Table 2 Addition modulo 52 on B (52)

$\oplus_{52}$	5	11	23	29	41	47
5	10	16	28	34	46	0
11	16	22	34	40	0	6

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23	28	34	46	0	12	18
29	34	40	0	6	18	24
41	46	0	12	18	30	36
47	0	6	18	24	36	42

Let $\lambda(2n)$ = number of Goldbach primes associated with 2n. Unlike π that assigns to n the number of primes that are less than or equal to n [1], λ is not necessarily increasing as is evident from the following table.

Table 3 Number of Goldbach primes associated with 2n

2n	$\lambda(2n)$
8	2
10	3
12	2
. . .	. . .
88	8
90	18
92	8
. . .	. . .
6568	140
6570	404
6572	140
. .	. . .
9868	198
9870	632
9872	204
. . .	. . .
9996	510
9998	197
10000	254
. . .	. . .

B (2n) may be written as

$B(2n)=\{a_1, a_2, a_3, \ldots, a_{\lceil \lambda(2n)/2 \rceil}, a_{\lambda(2n)-(\lceil \lambda(2n)/2 \rceil -1)}, \ldots, a_{\lambda(2n)-2}, a_{\lambda(2n)-1}, a_{\lambda(2n)}\}$ where

$a_1 < a_2 < a_3 < \ldots < a_{\lceil \lambda(2n)/2 \rceil} < a_{\lambda(2n)-(\lceil \lambda(2n)/2 \rceil -1)} < \ldots < a_{\lambda(2n)-2} < a_{\lambda(2n)-1} < a_{\lambda(2n)}$

and

$(a_1, a_{\lambda(2n)}), (a_2, a_{\lambda(2n)-1}), (a_3, a_{\lambda(2n)-2}), \ldots, (a_{\lceil \lambda(2n)/2 \rceil}, a_{s-(\lceil \lambda(2n)/2 \rceil -1)})$

are Goldbach pairs. Here $\lceil \ldots \rceil$ stands for ceiling function. We have

Proposition 3: $\lambda(2n)$ is odd iff n is prime.

Proof: Suppose n is prime then n and 2n – n are prime.
Therefore $n \in B(2n)$ and (n , n) is a Goldbach pair. If there are m Goldbach pairs for 2n, then $\lambda(2n) = 2(m-1) + 1 = 2m - 1$.

Conversely if $\lambda(2n)$ is odd then

$\lceil \lambda(2n)/2 \rceil = \lambda(2n) - (\lceil \lambda(2n)/2 \rceil - 1)$

and hence the first and second components of the Goldbach pair

$(a_{\lceil \lambda(2n)/2 \rceil}, a_{\lambda(2n)-(\lceil \lambda(2n)/2 \rceil -1)})$

are equal, while

$a_{\lceil \lambda(2n)/2 \rceil} + a_{\lambda(2n)-(\lceil \lambda(2n)/2 \rceil-1)} = 2n$

Therefore

$a_{\lceil \lambda(2n)/2 \rceil} = a_{\lambda(2n)-(\lceil \lambda(2n)/2 \rceil-1)} = n$

Hence n is prime. □

Proposition 4: If $p \in B(2n)$ then $q \in B(2n + (q - p))$ where p and q are primes.

Proof: Obvious because

$$p \in B(2n) \Rightarrow 2n - p \text{ is prime}$$
$$\Rightarrow 2n + (q - p) - q \text{ is prime}$$
$$\Rightarrow q \in B(2n + (q - p))$$ □

In particular if $3 \in B(2n)$ then $5 \in B(2n+2)$, $7 \in B(2n+4)$, $11 \in B(2n+8)$, . . . For example since $3 \in B(5090)$ therefore $5 \in B(5092)$, $7 \in B(5094)$, $11 \in B(5098)$, $13 \in B(5100)$, $17 \in B(5104)$, $19 \in B(5106)$. . .

Primes have very peculiar behaviour. The number $\lambda(2n)$ which represents the number of Goldbach primes associated with 2n behaves indifferently as well. However a nice thing about Goldbach primes associated with 2n is that they are evenly distributed on both sides of n. Goldbach conjecture asks for existence of such primes and that $\lambda(2n)$ is non-zero.

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