AN INTERESTING IDENTITY OF TWO INTEGRALS APPEARING IN REPRESENTATION THEORY

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Abstract

A certain UIR matrix element of SO(2, 1), when evaluated by two different methods, leads to two different integral expressions for it. The identity of these two expressions is established by using appropriate changes of variables of integration.

Keywords: Matrix Elements, Integrals, Change of variables.

INTRODUCTION

In the representation theory on non-compact rotation group SO(2, 1), a certain unitary irreducible representation matrix element [1], when evaluated by two completely different methods, leads to two different integral expressions for it viz.

$$\Im^{\sigma}{}_{\tau'\tau}(p',p,\zeta) \quad \text{and} \quad \{\Im^{\sigma}{}_{\tau\tau'}(p,p',-\zeta)\}^{\times}\ ,\ \zeta<0\ ,$$

Where

i) for $\zeta > 0$, [2],

$$\Im^{\sigma}{}_{\tau'\tau}(p',p,\zeta)$$

$$= \frac{1}{4\pi}\Big[\int_{|\phi|<\phi_0(\zeta)} d\phi \exp(-ip'\phi)\,|ch\zeta - ch\phi sh\zeta|^{-1/2+i\rho}\,\Big|\frac{e^{\phi}-\tanh\zeta/2}{1-e^{\phi}\tanh\zeta/2}\Big|^{ip}$$

$$+\ \tau' \int_{|\phi|>\phi_0(\zeta)} d\phi \exp(-ip'\phi)\,|ch\zeta - ch\phi sh\zeta|^{-1/2+i\rho}\Big|\frac{e^{\phi}-\tanh\zeta/2}{1-e^{\phi}\tanh\zeta/2}\Big|^{ip}$$

$$+\ \tau\tau' \int_{-\infty}^{\infty} d\phi \exp(-ip'\phi)\,|ch\zeta + ch\phi sh\zeta)^{-1/2+i\rho}\,\Big|\frac{e^{\phi}+\tanh\zeta/2}{1+e^{\phi}\tanh\zeta/2}\Big|^{ip}\Big]$$

ii) for $\zeta < 0$, [2] ,

$$\Im^{\sigma}{}_{\tau'\tau}(p',p,\zeta)$$

$$= \frac{1}{4\pi}\Big[\int_{-\infty}^{\infty} d\phi \exp(-ip'\phi)\,|ch\zeta - ch\phi sh\zeta|^{-1/2+i\rho}\Big|\frac{e^{\phi}-\tanh\zeta/2}{1-e^{\phi}\tanh\zeta/2}\Big|^{i\rho}$$

$$+\ \tau \int_{|\phi|>\phi_0(-\zeta)} d\phi \exp(-ip'\phi)\,|ch\zeta + ch\phi sh\zeta|^{-1/2+i\rho}\,\Big|\frac{e^{\phi}+\tanh\zeta/2}{1+e^{\phi}\tanh\zeta/2}\Big|^{ip}$$

$$+\ \tau\tau' \int_{|\phi|<\phi_0(-\zeta)} d\phi \exp(-ip'\phi)\,|ch\zeta + ch\phi sh\zeta|^{-1/2+i\rho}\,\Big|\frac{e^{\phi}+\tanh\zeta/2}{1+e^{\phi}\tanh\zeta/2}\Big|^{ip}\Big]$$

iii) $\sigma = -1/2 + i\rho$,

iv) $\phi_0(\zeta), \zeta > 0$ is defined by $\tanh\dfrac{\phi_0(\zeta/2)}{2} = e^{-\zeta}$,

v) τ, $\tau^{/}$ take the values +, - ,

vi) p, $p^{/}$ and ζ are real numbers.

The purpose of the present paper is to directly establish the identity of these two expressions by carrying out appropriate changes of variables of integration appearing in them.

PROOF OF THE IDENTITY

From (i) above, we will have, for $\zeta < 0$,

$$\{ F^{\sigma}_{\tau\tau^{/}}(p, p^{/}, -\zeta)\}^{\times}$$

$$= \frac{1}{4\pi}[\int_{|\phi|<\phi_0(-\zeta)} d\phi \exp(ip\phi) | ch\zeta + ch\phi sh\zeta |^{-1/2-i\rho} | \frac{e^{\phi} + \tanh\zeta/2}{1 + e^{\phi}\tanh\zeta/2} |^{-ip^{/}}$$

$$\tau \int_{|\phi|>\phi_0(-\zeta)} d\phi \exp(ip\phi) | ch\zeta + ch\phi sh\zeta |^{-1/2-i\rho} | \frac{e^{\phi} + \tanh\zeta/2}{1 + e^{\phi}\tanh\zeta/2} |^{-ip}$$

$$\tau\tau^{/} \int_{-\infty}^{\infty} d\phi \exp(ip\phi) | ch\zeta - ch\phi sh\zeta |^{-1/2-i\rho} | \frac{e^{\phi} - \tanh\zeta/2}{1 - e^{\phi}\tanh\zeta/2} |^{-ip}]$$

$$= \frac{(1-t^2)^{1/2+i\rho}}{4\pi}[\int_{-\phi_0(-\zeta)}^{\phi_0(-\zeta)} d\phi \exp(ip\phi) | \frac{(e^{\phi}-t)(1-te^{\phi})}{e^{\phi}} |^{-1/2-i\rho} | \frac{e^{\phi}-t}{1-te^{\phi}} |^{-ip^{/}}$$

$$+ \tau \int_{-\infty}^{-\phi_0(-\zeta)} d\phi \exp(ip\phi) | \frac{(e^{\phi}-t)(1-te^{\phi})}{e^{\phi}} |^{-1/2-i\rho} | \frac{e^{\phi}-t}{1-te^{\phi}} |^{-ip^{/}}$$

$$+ \tau \int_{\phi_0(-\zeta)}^{\infty} d\phi \exp(ip\phi) | \frac{(e^{\phi}-t)(1-te^{\phi})}{e^{\phi}} |^{-1/2-i\rho} | \frac{e^{\phi}-t}{1-te^{\phi}} |^{-ip^{/}}$$

$$+ \tau\tau^{/} \int_{-\infty}^{\infty} d\phi \exp(ip\phi) | \frac{(e^{\phi}+t)(1+te^{\phi})}{e^{\phi}} |^{-1/2-i\rho} | \frac{e^{\phi}+t}{1+te^{\phi}} |^{-ip^{/}}]$$

where $t = -\tanh\zeta/2 > 0$, and we have used the easily verifiable facts that

$$ch\zeta + ch\phi sh\zeta = \frac{e^{-\phi}}{1-t^2}(e^{\phi} - t)(1 - te^{\phi}) \ ,$$

$$ch\zeta - ch\phi sh\zeta = \frac{e^{-\phi}}{1-t^2}(e^{\phi} + t)(1 + te^{\phi}) \ .$$

Calling e^{φ} as x, we will have

dx = $e^{\phi} d\phi \Rightarrow d\phi = \frac{dx}{x}$,

$$\phi < \phi_0(-\zeta) \Rightarrow e^{\phi} < e^{\phi_0(-\zeta)} = -\frac{1}{\tanh \zeta / 2} = \frac{1}{t} \Rightarrow x < \frac{1}{t},$$

$$\phi > \phi_0(-\zeta) \Rightarrow x > \frac{1}{t},$$

$$\phi > -\phi_0(-\zeta) \Rightarrow e^{\phi} > e^{-\phi_0(-\zeta)} = t \Rightarrow x > t,$$

$$\phi < -\phi_0(-\zeta) \Rightarrow x < t,$$

$$\phi > -\infty \Rightarrow e^{\phi} > 0 \Rightarrow x > 0,$$

$$\phi < \infty \Rightarrow e^{\phi} < \infty \Rightarrow x < \infty,$$

so that the above expression will become

$$\frac{(1-t^2)^{1/2+i\rho}}{4\pi} \times$$

$$[\int_t^{1/t} \frac{dx}{x} x^{ip} (\frac{(x-t)(1-tx)}{x})^{-1/2-i\rho} (\frac{x-t}{1-tx})^{-ip'}$$

$$+ \ \tau \int_0^t \frac{dx}{x} x^{ip} (\frac{(t-x)(1-tx)}{x})^{-1/2-i\rho} (\frac{t-x}{1-tx})^{-ip'}$$

$$+ \ \tau \int_{1/t}^{\infty} \frac{dx}{x} x^{ip} (\frac{(x-t)(tx-1)}{x})^{-1/2-i\rho} (\frac{x-t}{tx-1})^{ip'}$$

$$+ \ \tau\tau' \int_0^{\infty} \frac{dx}{x} x^{ip} (\frac{(x+t)(1+tx)}{x})^{-1/2-i\rho} (\frac{x+t}{1+tx})^{ip'}] \ .$$

We now make a change of variable of integration from x to x', where

i) in the first integral, with $t < x < 1/t$, we put

$$x' = \frac{x-t}{1-tx}$$

$$\Rightarrow x = \frac{x'+t}{1+tx'}, \quad 1-tx = \frac{1-t^2}{1+tx'}, \quad \frac{x-t}{x} = (1-t^2)\frac{x'}{x'+t},$$

$$dx = (1-t^2)\frac{dx'}{1+tx'}, \quad x = t \rightarrow x' = 0, \quad x = \frac{1}{t} \rightarrow x' \rightarrow \infty,$$

ii) in the second integral, with $0 < x < t$, we put

$$x' = \frac{t-x}{1-tx}$$

$$\Rightarrow x = \frac{t-x'}{1-tx'}, \quad 1-tx = \frac{1-t^2}{1-tx'}, \quad \frac{t-x}{x} = (1-t^2)\frac{x'}{t-x'},$$

$$dx = -\frac{1-t^2}{(1-tx')^2}, \quad x = 0 \rightarrow x' = t, \quad x = t \rightarrow x' = 0,$$

iii) in the third integral, with $1/t < x < \infty$, we put

$$x' = \frac{x-t}{tx-1}$$

$$\Rightarrow x = \frac{x'-t}{tx-1}, \quad tx-1 = \frac{1-t^2}{tx'-1}, \quad \frac{t-x}{x} = (1-t^2)\frac{x'}{x'-t},$$

$$dx = -\frac{1-t^2}{(tx'-1)^2}dx', \quad x = 1/t \rightarrow x' \rightarrow \infty, \quad x \rightarrow \infty \rightarrow x' = 1/t,$$

iv) in the fourth integral, with $0 < x < \infty$, we put

$$x' = \frac{x+t}{1+tx}$$

$$\Rightarrow x = \frac{x^{/} - t}{1 - tx^{/}} \ , \quad 1 + tx = \frac{1 - t^2}{1 - tx^{/}} \ , \quad \frac{x + t}{x} = (1 - t^2)\frac{x^{/}}{x^{/} - t} \ ,$$

$$dx = \frac{1 - t^2}{(1 - tx^{/})^2} dx^{/} \ , \quad x = 0 \ \rightarrow x^{/} = t \ , \quad x \rightarrow \infty \rightarrow x^{/} = 1/t \ .$$

Then the above expression will then transform to

$$\{ F_{\tau\tau^{/}}^{\sigma}(p, p^{/}, -\zeta)\}^{\times} = \frac{(1 - t^2)^{1/2 + i\rho}}{4\pi} \times$$

$$[\int_0^{\infty} dx^{/} \frac{1 - t^2}{(1 + tx^{/})^2} (\frac{x^{/} + t}{1 + tx^{/}})^{ip - 1} ((1 - t^2)\frac{x^{/}}{x^{/} + t} . \frac{1 - t^2}{1 + tx^{/}})^{-1/2 - i\rho} (x^{/})^{-ip^{/}}$$

$$+ \tau \int_t^0 dx^{/} . - \frac{1 - t^2}{(1 - tx^{/})^2} (\frac{t - x^{/}}{1 - tx^{/}})^{ip - 1} ((1 - t^2)\frac{x^{/}}{t - x^{/}} . \frac{1 - t^2}{1 - tx^{/}})^{-1/2 - i\rho} (x^{/})^{-ip^{/}}$$

$$+ \tau \int_{\infty}^{1/t} dx^{/} . - \frac{1 - t^2}{(tx^{/} - 1)^2} (\frac{x^{/} - t}{tx^{/} - 1})^{ip - 1} ((1 - t^2)\frac{x^{/}}{x^{/} - t} . \frac{1 - t^2}{tx^{/} - 1})^{-1/2 - i\rho} (x^{/})^{-ip^{/}}$$

$$+ \tau\tau^{/} \int_t^{1/t} dx^{/} \frac{1 - t^2}{(1 - tx^{/})^2} (\frac{x^{/} - t}{1 - tx^{/}})^{ip - 1} ((1 - t^2)\frac{x^{/}}{x^{/} - t} . \frac{1 - t^2}{1 - tx^{/}})^{-1/2 - i\rho} (x^{/})^{ip^{/}}]$$

$$= \frac{(1 - t^2)^{1/2 - i\rho}}{4\pi} \times$$

$$[\int_0^{\infty} \frac{dx^{/}}{x^{/}} (x^{/})^{-ip^{/}} \ | \frac{(x^{/} + t)(1 + tx^{/})}{x^{/}} |^{-1/2 + i\rho} | \frac{x^{/} + t}{1 + tx^{/}} |^{ip}$$

$$+ \tau \int_0^t \frac{dx^{/}}{x^{/}} (x^{/})^{-ip^{/}} \ | \frac{(t - x^{/})(1 - tx^{/})}{x^{/}} |^{-1/2 + i\rho} | \frac{t - x^{/}}{1 - tx^{/}} |^{ip}$$

$$+ \tau \int_{1/t}^{\infty} \frac{dx^{/}}{x^{/}} (x^{/})^{-ip^{/}} \ | \frac{(x^{/} - t)(tx^{/} - 1)}{x^{/}} |^{-1/2 + i\rho} | \frac{x^{/} - t}{tx^{/} - 1} |^{ip}$$

$$+ \tau\tau' \int_{t}^{1/t} \frac{dx'}{x'} (x')^{-ip'} \left| \frac{(x'-t)(1-tx')}{x'} \right|^{-1/2+i\rho} \left| \frac{x'-t}{1-tx'} \right|^{ip}]$$

$$= \frac{1}{4\pi} [\int_{-\infty}^{\infty} d\phi \exp(-ip'\phi) \, | ch\zeta - ch\phi sh\zeta |^{\sigma} \left| \frac{e^{\phi} - \tanh\zeta/2}{1 - e^{\phi} \tanh\zeta/2} \right|^{ip}$$

$$+ \tau \int_{|\phi| > \phi_0(-\zeta)} d\phi \exp(-ip'\phi) \, | ch\zeta + ch\phi sh\zeta |^{\sigma} \left| \frac{e^{\phi} + \tanh\zeta/2}{1 + e^{\phi} \tanh\zeta/2} \right|^{ip}$$

$$+ \tau\tau' \int_{|\phi| < \phi_0(-\zeta)} d\phi \exp(-ip'\phi) \, | ch\zeta + ch\phi sh\zeta |^{\sigma} \left| \frac{e^{\phi} + \tanh\zeta/2}{1 + e^{\phi} \tanh\zeta/2} \right|^{ip}]$$

(where now $x' = e^{\phi}$) which is identical with the explicit expression for

$$F^{\sigma}_{\tau'\tau}(p', p, \zeta) , \qquad \zeta < 0 ,$$

given in the Introduction.

CONCLUSION

We conclude that the identity of the two integral expressions for the relevant matrix element mentioned in the Introduction, can be established, but for this, we need to use four different transformations of the variables of integration in the four integrals appearing in the expression for the matrix element.

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