QUANTALOIDS AND SHEAVES ON RIGHT GELFAND QUANTALES

Muhammad Nawaz
(nawaz@buitms.edu.pk)
Balochistan University of Information Technology and Management Sciences, Samungli Road, Jinnah Town, Quetta, Pakistan

Abstract

A quantale is a semigroup in *SUP*. A right Gelfand quantale is a complete lattice equipped with an associative binary operation & (possibly non-commutative) which is idempotent, has right identity and distributes over arbitrary suprema on both sides. A quantaloid is a *SUP*-enriched category. In this paper we obtain a quantaloid from a right Gelfand quantale. We also discuss two approaches for introduction of concept of sheaf on a right Gelfand quantale Q. In both cases the construction coincides with the concept of sheaf in case Q is a locale.

AMS Mathematics Subject Classification (2000): 06F07, 18B35, 18D05, 18D20

Keywords: *Topos, Grothendieck Topos, Bicategory, Quantales, Quantaloids, Sheaf.*

Corresponding Authos's email: nawaz@buitms.edu.pk

INTRODUCTION

As introduced in [4], a quantale is a semigroup in *SUP* the category of sup-lattices and sup-preserving maps. In other words, a quantale is a complete lattice equipped with an associative product which is distributive on both sides over arbitrary suprema. A right Gelfand quantale [3] is a complete lattice Q equipped with an associative binary operation (possibly non-commutative) & which is idempotent, has right identity and distributes over arbitrary suprema on both sides.

Let A be a unital C^* algebra, then the closed linear subspaces of A form a quantale where suprema of a family of closed linear subspaces is given by taking the closure of algebraic sum, and the product & of any two closed linear subspaces of A is given by taking the closure of their algebraic product. Let us write this quantale as *Max A*. Those elements of *Max A* on which the top element (of *Max A*) acts as a unit on the right, are exactly the closed right ideals of A and again form a quantale with respect to operations of *Max A*. For any closed right ideal I of A, we have $I \,\&\, I = I$. Hence the closed right ideals of a C^*- algebra A form an idempotent quantale. In this quantale, the top element acts as unit from right. Hence the closed right ideals of a C^*- algebra A form a right Gelfand quantale. Similarly the lattice of closed left ideals of a C^* algebra is a quantale with respect to multiplication of left ideals; again the top element of the lattice is unit from left hand side.

A quantaloid [6] is a category whose hom-sets are sup-lattices in which composition distributes on both sides over arbitrary suprema of morphisms [7], thus a quantaloid is a *SUP*-enriched category. For any object a of a quantaloid, the hom-set $hom(a, a)$ is a quantale [6], hence a quantaloid with one object is a quantale as well. Simplest example of quantaloid is the category *SUP*. The category of relations in a Grothendieck topos is also a quantaloid [7]. Given a ring R (possibly non-commutative) with unit, a quantaloid with two objects 0 and 1 may be defined as having following hom-sets; $hom\ (0,\ 0)$ = additive subgroups of R, $hom\ (0,\ 1)$ = sup-lattice of left ideals of R, $hom(1,\ 0)$ = Sup-lattice of right ideals of R and $hom(1,\ 1)$ = sup-lattice of two-sided ideals of R. The identity arrows on 1 and 0 are R and the center $Z(R)$ of R respectively; details may be seen in [7, 8]. Another example of quantaloid is *Relations(H)* [12] for which objects are the elements of H and the hom-lattices are given by $hom\ (u, v) = \{ w \in H \,/\, w \leq u \wedge v\}$ where H is a locale.

We begin the paper with definition of right Gelfand quantale.

Right Gelfand Quantales

A right Gelfand quantale [3] is a complete lattice Q equipped with a binary operation (possibly non-commutative) &: $Q \times Q \rightarrow Q$ satisfying

i. $p \,\&\, \bigvee_I q_i = \bigvee_I (p \,\&\, q_i)$, $(\bigvee_I q_i) \,\&\, p = \bigvee_I (q_i \,\&\, p)$

ii. $(p \,\&\, q) \,\&\, r = p \,\&(q \,\&\, r)$

iii. $p \,\&\, p = p$

iv. $p \,\&\, 1 = p$

where 1 is the top element of Q.

From this definition, it is clear that $p \,\&\, q \leq p$ and p& q & r = p & r & q for all p, q, r $\in Q$.

Proposition: For a right Gelfand quantale Q and $p, q, r \in Q$,

$$r \leq p \,\&\, q \;\Rightarrow\; r = r \,\&\, q$$

Proof: $r \leq p \,\&\, q \;\Rightarrow\; r \leq 1 \,\&\, q$

Therefore $r \leq r \,\&\, q$ because of idempotency

But $r \,\&\, q \leq r$, hence $r = r \,\&\, q$.

From a right Gelfand quantale Q we obtain a category ϑ as follows.

Ob ϑ: Elements of Q.

$\vartheta\ (p\ , q) : \{r \in Q \mid r \leq p \,\&\, q, r = p \,\&\, r\}$

The identity arrow of $\vartheta\ (p, p)$ is p itself, we denote it by I_p.

Composition: For $p, q, r \in Q$

$$\vartheta(p, q) \times \vartheta(q, r) \xrightarrow{C(p,q,r)} \vartheta(p, r)$$

$$(u, v) \mapsto u \,\&\, v$$

Left identity for this composition is consequence of the condition

$r \leq p \,\&\, q$, $r = p \,\&\, r$ imposed on the elements of $\vartheta(p, q)$ and the right identity follows from the fact that $r \leq p \,\&\, q \Rightarrow r = r \,\&\, q$, that is, the following diagrams commute.

$$\begin{tikzcd} 1 \times \vartheta(p, q) \arrow[r, "I_p \times I_{\vartheta(p,q)}"] \arrow[dr] & \vartheta(p, p) \times \vartheta(p, q) \arrow[d, "C(p, p, q)"] \\ & \vartheta(p, q) \end{tikzcd} \qquad \begin{tikzcd} \vartheta(p, q) \times 1 \arrow[r, "I_{\vartheta(p,q)} \times I_q"] \arrow[dr] & \vartheta(p, q) \times \vartheta(q, q) \arrow[d, "C(p, q, q)"] \\ & \vartheta(p, q) \end{tikzcd}$$

Proposition: The category ϑ obtained above is a quantaloid.

Proof: It only remains to prove that $\vartheta(p, q)$ is a sup- lattice for all $p, q \in Q$.

Let $\{s \in Q \mid s \leq p \,\&\, q, s = p \,\&\, s\}$ be a sub-family of $\vartheta(p, q)$ then

$\bigvee s \leq p \,\&\, q$. Also $s = p \,\&\, s \Rightarrow \bigvee s = \bigvee (p \,\&\, s) = p \,\&\, \bigvee s$.

Therefore $\vartheta(p, q)$ is a (complete) sup-lattice. Hence $\vartheta(p, q)$ is a quantaloid.

Sheaves

A poset (partially ordered set) may be considered as a category. Hence ϑ(p, q) is a category for every pair p, q $\in Q$. This makes ϑ a bicategory [1] in which

Objects: Elements of Q.

Arrows: Elements $r \in Q$ such that $r \leq p \,\&\, q$, $r = p \,\&\, r$

2-Cells: order in Q.

Composition of 2-cells is given as follows

$$\vartheta(p, q) \times \vartheta(q, r) \xrightarrow{C(p,q,r)} \vartheta(p, r)$$

$$(u \leq u', v \leq v') \mapsto u \,\&\, v \leq u' \,\&\, v'$$

Diagrammatically, this requires that

$$\begin{tikzcd}
\vartheta(p,q)\times\vartheta(q,r)\times\vartheta(r,s) \arrow[r, "I_{\vartheta(p,q)}\times C(q,r,s)"] \arrow[d, "C(p,q,r)\times I_{\vartheta(r,s)}"'] & \vartheta(p,q)\times\vartheta(q,s) \arrow[d, "C(p,q,s)"] \\
\vartheta(p,r)\times\vartheta(r,s) \arrow[r, "C(p,r,s)"'] & \vartheta(p,s)
\end{tikzcd}$$

Commutes.

Theory of categories enriched in a bicategory is being intensively studied these days [7-11]. Since ϑ is a bicategory, we define a ϑ-category as:

A ϑ-category is a set X together with functions

$e : X \to \mathrm{Ob}(\vartheta)$

$d: X\times X \to \mathrm{Mor}(\vartheta)$

satisfying

i. $d(x_1, x_2) : e(x_1) \to e(x_2)$

ii. $I_{e(x)} \leq d(x, x)$

iii. $d(x_2, x_3) \circ d(x_1, x_2) \leq d(x_1, x_3)$

A ϑ-functor $F: X \to Y$ is a function satisfying

i. $e(F(x)) = e(x)$

ii. $d(x_1, x_2) \leq d(F(x_1), F(x_2))$ for all $x_1, x_2 \in X$.

Further, notions of presheaf and sheaf on Q may be introduced as follows [3]

Sheaves on Q

A presheaf on Q is a set A together with mappings

$E: A \to Q$, $\downharpoonleft : Q \times A \to A$

satisfying

i. $(E\,a) \downharpoonleft a = a$

ii. $P \downharpoonleft (q \downharpoonleft a) = (p \,\& q) \downharpoonleft a$

iii. $E(p \downharpoonleft a) = p \,\&\, Ea$.

A morphism of presheaves is a mapping $f: A \to B$ satisfying

i. $Ea = Ef(a)$

ii. $f(p{\restriction}a) = p{\restriction}f(a)$.

E and $\restriction$ induce a partial order $\prec$ on underlying set of a presheaf A given by

$$a \prec a' \Leftrightarrow Ea \leq Ea' \text{ and } a = Ea{\restriction}a'.$$

A presheaf A is a <u>sheaf</u> if every compatible family has a unique join where

$B \subseteq A$ is compatible if $Eb{\restriction}b' = Eb \,\&\, Eb'{\restriction}b$ for all $b, b' \in B$.

It can be shown that the category of sheaves on Q and the category of ϑ-categories (satisfying certain conditions) are isomorphic. Details will appear elsewhere [5].

Another approach for sheaf on Q

We now discuss another approach for introduction of notion of sheaf on a Gelfand quantale due to Francis Borceux [2]. Both notions (the one given above and the one which follows) reduce to the concept of sheaf on a locale in familiar fashion in case Q happens to be a locale.

We note that the assignment $\downarrow : Q \longrightarrow SET$, given by

$$p \longmapsto \downarrow p = \{u \,/\, u \leq p\} \qquad \text{(on objects)}$$

$$p \leq q \longmapsto \downarrow q \longrightarrow \downarrow p;\ u \longmapsto p \,\&\, u \qquad \text{(on morphisms)}$$

is not functorial because for p = q and u ≤ p, we may not have p & u = u in general. Thus the identity morphisms in **Q** are not preserved. In order to make this assignment functorial, we add a new morphism $p \xrightarrow{\rho} q$ whenever p ≤ q and for u ≤ q it is the case that u & p ≤ p. This makes

$$\downarrow q \longrightarrow \downarrow p;\ u \longmapsto u \,\&\, p \qquad \text{(on morphisms)}$$

well defined.

Further, morphism ρ always exists from p to p because for u ≤ p we always have u & p ≤ p and also u = u & p (u & p ≤ u, also u ≤ p gives u ≤ u & p).

Thus we obtain a category ζ whose objects are the elements of Q, morphisms are as defined above.

Now compositions may be defined in the following way for which $p \xrightarrow{\rho} p$ works as identity morphism.

$$p \xrightarrow{\rho} q \xrightarrow{\lambda} r = p \xrightarrow{\lambda} r$$

$$p \xrightarrow{\lambda} q \xrightarrow{\rho} r = p \xrightarrow{\lambda} r$$

$$p \xrightarrow{\rho} q \xrightarrow{\rho} r = p \xrightarrow{\rho} r$$

$$p \xrightarrow{\lambda} q \xrightarrow{\lambda} r = p \xrightarrow{\lambda} r$$

Further for any p ∈ **Q** we require that

$$p \xrightarrow{\lambda} p = p \xrightarrow{\rho} p$$

if and only if p & r = r for all r ≤ p. We can then define a Grothendieck topology J on ζ as

$$R \in J(p) \iff p = \bigvee \{ q \in \mathbf{Q} \mid (q \xrightarrow{\rho} p) \in R \}.$$

A sheaf on **Q** in the sense of [2] is a sheaf on the site (J, ζ).

REFERENCES

1. Benabou J. (1967). Introduction to bicategoires, “*Reports of the Midwest Category seminar*”, Lecture notes in Mathematics, springer, Berlin.

2. Borecux F and Van Den Bossche G. (1986). *Quantales and their sheaves*. 3:61.

3. Mulvey CJ and Nawaz M. (1986). *Quantales: Quantal sets*, Theory and Decision Library, Series B: Mathematical and Statistical Methods Volume 32, Non-Classical Logics and their Applications to Fuzzy Subsets, Kluwer Academic Publishers, Netherlands. 159-217, 1995

4. Mulvey CJ and Suppl. (1986). Rend. Circ. Mat. Palermo 12: 99-104

5. Nawaz M. (2007). *Quasisymmetric ℩B–categories*, Demonstratio Mathematica, 40 (3): 517-527.

6. Rosenthal KI. (1996). *The theory of quantaloids*, Pitman Research Notes in Mathematics Series. Longman, Harlow 1996.

7. Stubbe I. (2003)., *Categorical structures enriched in a quantaloid: categories and semicategories*, thesis, Universite de Catholique de Louvain.

8. Stubbe I. (2005). *Categorical structures enriched in a quantaloid: categories, distributors and functors*, Theory and Applications of Categories. 14:1-45.

9. Stubbe I. (2006). *Categorical structures enriched in a quantaloid: tensored and cotensored categories*, Theory and Applications of Categories. 16:283-306.

10. Stubbe I. (2005). *Categorical structures enriched in a quantaloid: regular presheaves, regular semicategories*, Cahiers de Topologie et Geometrie Differentielle Categoriques. 46:99-121.

11. Stubbe I. (2005). *Categorical structures enriched in a quantaloid: orders and ideals over a base quantaloid*, Applied Categorical Structures. 13:235-255

12. Walters RFC. (1981). *Sheaves and Cauchy-complete Categories*, Cahiers de Topologie et Geometrie Differentielle. 22.3:283-286.