J. App. Em. Sc: 1(1) 143 – 151, June 2004

Localic Reflections of Quantales

M. Nawaz (nawaz@buitms.edu.pk)
Balochistan University of Information Technology
and Management Sciences Quetta

Abstract

Quantales were introduced by C.J Mulvey (2) to provide a non-commutative extension of the concept of locale (1). Certain authors (4) restrict the term quantale to mean those multiplicative lattices for which each element is right sided ($a \& 1 < a$) and idempotent ($a \& a = a$). In this paper by a quantale we mean a complete multiplicative lattice with $1 = \top$. The paper deals with localic reflections of quantales. A condition for neat elements of a coherent quantale to form a locale has also been studied.

1- Basic Concepts and definitions

1.1 A <u>poset</u> (partially ordered set) is a set A equipped with a binary relation (partial order) $\leq$ which is

reflexive : $a \leq a$

transitive : if $a \leq b$ and $b \leq c$ then $a \leq c$

anti-symmetric: if $a \leq b$ and $b \leq a$ then $a = b$.

1.2 Let A be a poset and $a, b \in A$, then $a \vee b$ is called the <u>join</u> of a and b if

i) $a \leq a \vee b$ and $b \leq a \vee b$

ii) $a \leq c$ and $b \leq c \Rightarrow a \vee b \leq c$.

For $S \subseteq A$, we write $\bigvee S$ for join of S, that is

i) $s \leq \bigvee S$ for all $s \in S$

ii) If $s \leq b$ for all $s \in S$ then $\bigvee S \leq b$

In any poset A, we introduce <u>meet</u> by reversing all the inequalities in the definition of join. For $a, b \in A$, $a \wedge b$ stands for meet of a and b and $\wedge S$ for the meet of S for any $S \subseteq A$.

1.3 A <u>lattice</u> is a poset in which every finite subset has both a join and a meet. Meet and join of empty subset are denoted by $\top$ and $\bot$ respectively. $\top$ is the greatest and $\bot$ is the least element of the lattice.

A lattice A is distributive if

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

for all $a, b, c \in A$.

A <u>boolean algebra</u> is a distributive lattice A equipped with unary operation

$$\neg : A \rightarrow A$$

such that

$$(\neg a) \wedge a = \bot$$

and

$(\neg a) \vee a = \top$.

1.4 A lattice A is called complete if every subset of A has a join in A.

A locale is a complete lattice L satisfying the infinite distributive law

$$a \wedge \bigvee_i b_i = \bigvee_i (a \wedge b_i)$$

Terms "frame" and "complete hyeting algebra" are also used for locales, however, locales, frames and hyeting algebras have different meaning when their categories are considered.

2- Multiplicative Lattices and Quantales

Definitions

2.1 A complete lattice L together with $\& : L \times L \rightarrow L$ satisfying

i) & is associative

ii) $a \,\&\, \bigvee b_i = \bigvee (a \,\&\, b_i)$ and $(\bigvee b_i) \,\&\, a = \bigvee (b_i \,\&\, a)$

iii) $b \leq c \Rightarrow a \,\&\, b \leq a \,\&\, c$ and $b \,\&\, a \leq c \,\&\, a$

$\forall\, a, b, c \in L$, $\{ b_i \,/\, i \in I \} \subseteq L$, is called a complete multiplicative lattice.

2.2 $1 \in L$ is called multiplicative identity if

$1 \,\&\, a = a \,\&\, 1 = a \qquad \forall a \in L$

2.3 $1 \in L$ is called right (left) multiplicative identity if

$a \,\&\, 1 = a, \qquad (1 \,\&\, a = a)$.

2.4 The term identity will be used for $1 \in L$ if

$1 = \top$ and $a \,\&\, 1 = 1 \,\&\, a = a \quad \forall a \in L$

2.5 $1 = \top$ is right (left) identity if $a \,\&\, 1 = a \;\; (1 \,\&\, a = a)$.

2.6 L is commutative if $a \,\&\, b = b \,\&\, a \qquad \forall a, b \in L$

2.7 L is idempotent if $a \,\&\, a = a \qquad \forall a \in L$

Examples:

i) Lattice of right (left) ideals of a ring (with 1) is non-commutative complete multiplicative lattice with right (left) identity. RIdl(A) is idempotent if A is regular ring. For commutative ring A, Idl A is commutative. Two sided ideals of a non-commutative ring form complete multiplicative lattice with identity. Lattice of neat ideals (1) is locale. Another example of locale is the lattice Rad A of redical ideals of a commutative ring A.

ii) Opens of a topological group is complete multiplicative lattice, without multiplicative identity in general. O(G) has multiplicative identity iff G is discrete. O(G) is commutative if G is.

iii) Every locale is commutative and idempotent complete multiplicative lattice with identity by putting $\& = \wedge$.

2.8 By a quantale Q we mean a complete multiplicative lattice with $1 = \top$

A morphism of quantles $f : L \to M$ is an & $-\bigvee$ preserving map. We denote by QtL, the category of quantales and & $-\bigvee$ preserving maps and by CQtL, the subcategory of commutative quantales.

Remarks

For commutative rings A and B, any ring homomorphism $f : A \to B$ gives rise to a quantale homomorphism.

$$\underline{f} : \mathrm{Idl}(A) \to \mathrm{Idl}(B)$$
$$I \mapsto [\, f(I) \,]$$

Where [f(I)] is ideal generated by f(I). In fact we have a functor

$F : C\,\mathrm{Rng} \to \mathrm{CQtL}$ which takes any commutative ring R to IdlR and morphism

$f : R \to S$ to $\underline{f} : \mathrm{IdlR} \to \mathrm{IdlS}$.

Proposition 2.1(C.J. Mulvey)
Let A be a commutative ring, then the locale Rad A of radical ideals of A is the localic reflection of the quantale Idl A of ideals of A.

Proof:

Consider the mapping.

$$\sqrt{-} : \mathrm{Idl}\ A \to \mathrm{Rad}\ A$$

which assigns to each ideal of A its radical. It is asserted that this is & $\bigvee$ -morphism from the quantale Idl A to the quantle Rad A. Further that any such homomorphism factors uniquely through this one.

$$\sqrt{IJ} = \sqrt{I} \wedge \sqrt{J} :$$

Since

$$IJ \leq I \wedge J \leq \sqrt{I} \wedge \sqrt{J}$$

and $\sqrt{I} \wedge \sqrt{J}$ is radical ideal, therefore

$$\sqrt{IJ} \leq \sqrt{I} \wedge \sqrt{J}.$$

Conversely if $a \in \sqrt{I} \wedge \sqrt{J}$ then there exists n with $a^n \in I \wedge J$. Let $P \supseteq IJ$ where P is prime ideal, then

$$a^{2n} = a^n \cdot a^n \in IJ \subseteq P.$$

Hence $a \in P$.

So $a \in \sqrt{IJ}$, thus $\sqrt{I} \wedge \sqrt{J} \leq \sqrt{IJ}$.

b) $\sqrt{\Sigma I_i} = \sqrt{\Sigma \sqrt{I_i}}$:

$$\sqrt{\Sigma I_i} \leq P \Leftrightarrow \Sigma I_i \leq P$$
$$\Leftrightarrow I_i \leq P \text{ for each } i$$

$$\Leftrightarrow \sqrt{I_i} \leq P \text{ for each } i$$

$$\Leftrightarrow \sqrt{\Sigma\sqrt{I_i}} \leq P.$$

Thus $\sqrt{\ }$- is $\vee$ an & -morphism from Idl A to Rad A.

c) Consider

$\varphi : \text{Idl } A \rightarrow M$

where M is a locale and define

$\psi : \text{Rad } A \rightarrow M$

$I \mapsto \varphi(I)$

To see that ψ is $\vee$ an & -morphism, consider I, J $\in$ Rad A. Then

$$\psi(I \wedge J) = \varphi(I \wedge J)$$

which must be proved equal to $\varphi(I) \wedge \varphi(J)$. Assert that

$$\varphi(IJ) = \varphi(I \wedge J).$$

Clearly

$$\varphi(IJ) \leq \varphi(I \wedge J)$$

because $I\ J \leq I \wedge J$.

Conversely let $a \in I \wedge J$, then $a^2 \in IJ$. But

$(a)^2 = (a^2)$

hence

$\varphi(a^2) = \varphi(a)^2 = \varphi(a)$ (because M is a locale)

But

$$I \wedge J = \Sigma (a) \qquad (a \in I \wedge J)$$

$$\Rightarrow \varphi(I \vee J) = \varphi(a) \qquad (a \in I \wedge J)$$

$$\Rightarrow \varphi(I \vee J) = \varphi(a^2) \quad (a \in I \wedge J)$$

$$\leq \varphi(b)^{\vee} \qquad (b \in IJ) \text{ (since } a \in I \wedge J \Rightarrow a^2 \in IJ)$$

$$= \varphi(\Sigma(b)) \quad (b \in IJ)$$

$$= \varphi(IJ)$$

Thus

$$\varphi(I \wedge J) = \varphi(IJ) = \varphi(I) \wedge \varphi(J)$$

And therefore

$$\psi(I \wedge J) = \psi(I) \wedge \psi(J).$$

Now we will show that $\psi(\sqrt{\Sigma I_i}) = \psi(I_i)$:

First we note that

$$\varphi(\sqrt{I}) = \varphi(I)$$

for any ideal I, because

$I \leq \sqrt{I} \Rightarrow \varphi(I) \leq \varphi(\sqrt{I})$.

Conversely

$\sqrt{I} = \Sigma\,(a)$ $\quad (a^n \in I \text{ for some } n)$

$\Rightarrow \varphi\,(\sqrt{I}) = \bigvee \varphi(a)$ $\quad (a^n \in I)$

$\Rightarrow \varphi\,(\sqrt{I}) = \bigvee \varphi(a^{2n})$ $\quad (a^n \in I)$, since $\varphi(a) = \varphi(a^2) = \ldots = \varphi(a^{2n})$

$\Rightarrow \varphi\,(\sqrt{I}) \leq \bigvee \varphi(b)$ $\quad (b \in I)$

$= \varphi(\Sigma(b))$ $\quad (b \in I)$

$= \varphi(I)$

So $\quad \varphi\,(\sqrt{I}) \leq \varphi(I)$.

Therefore

$$\psi\,(\sqrt{\Sigma I_i}) = \overline{\varphi}\,(\sqrt{\Sigma I_i}) = \overline{\varphi}\,(\Sigma I_i) = \quad \varphi(I_i) = \bigvee \psi(I_i) \qquad \bigvee$$

as required.

Proposition 2.2

A quantale Q is localic iff $a^2 = a \ \ \forall a \in Q$ and Q has two sided identity.

Proof

$\Rightarrow$ is clear

$\Rightarrow$ Existence of two sided identity gives

$a \,\&\, b \leq a \wedge b$.

Now

$a \wedge b \leq a$ and $a \wedge b \leq b \Rightarrow (a \wedge b) \,\&\, (a \wedge b) \leq a \,\&\, b$

Idempotency gives

$a \wedge b \leq a \,\&\, b$

Thus

$a \,\&\, b = a \wedge b$

Hence & is same as $\wedge$. Therefore Q is a locale.

Proposition 2.3

Idempotents of any quantale Q with right (left) identity from a sup lattice.

Proof

$\perp$ and $1 = \top$ are idempotents.

Let S be any family of idempotents, then

$$\bigvee_{a_i \in S} \; \bigvee_{a_j \in S} a_i \,\&\, a_j \quad \bigvee_{a_i \in S} \bigvee_{a_j \in S} a_i \,\&\, a_j$$

$$= \bigvee_{a \in S} a \qquad \text{(Because } a_i \,\&\, a_j = a_i \text{ if } i = j,\ \leq a_i \text{ if } i \neq j\text{)}$$

Proposition 2.4:

For idempotent quantale Q with right (left) identity, for any $p \in Q$

$$\downarrow p = \{ a \in Q \mid a \leq p \}$$

is idempotent quantale with right (left) identity.

Proof

$\downarrow p$ is idempotent is clear.

Now $a \leq p$ and $b \leq p \Rightarrow a \,\&\, b \leq p$ (Because of idempotency).

Therefore $a \,\&\, b \in \downarrow p$.

Further for any $S \subseteq \downarrow p$ it is clea$\bigvee$that $S \leq p$.

p is right identity in $\downarrow p$ because

$a \leq p \Rightarrow a \,\&\, a \leq a \,\&\, p \leq a$

$\Rightarrow a \leq a \,\&\, p \leq a$

$\Rightarrow a \,\&\, p = a.$

The distributive law induces from Q.

Remarks

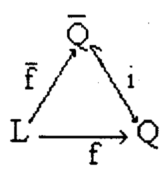
For commutative quantale, idempotents from a locale with $\& = \wedge$. For quantale Q we denote this locale by $\bar{Q}$

Any $\&\bigvee$-homomorphism $Q \xrightarrow{f} Q'$ preserves idempotents :

For any idempotent $a \in Q$, $f(a) = f(a \,\&\, a) = f(a) \,\&\, f(a)$.

Proposition 2.5

Given a commutative quantale Q, $\exists$ a locale such th$\bar{Q}$for any & preservin$\bigvee$map $L \rightarrow Q$, L a locale $\exists!$ L $\rightarrow$ such that $\bar{Q}$



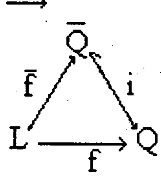
Commutes

Proof

First we note that any & preservi$\bigvee$g homomorphism Q Q′ prese$\xrightarrow{\varepsilon}$ idempotents.

Let $= \{ a \in \bar{Q} \mid a^2 = a \}$

The locale L is an idempotent quantale in particular, therefore from above observation we have for any L Q Imf $\subseteq$ $\subseteq$ Q w$\bar{Q}$ich gives such tha$\bar{f}$ $\xrightarrow{f}$



Commutes, is uniq$\bar{f}$ because i is inclusion.

We observed that for any quantale Q, Given $L \to$ we c$\overline{Q}$ always obtain a quantale homomorphism by composing with inclusion that is $L \to$ Q. $\overline{Q} \hookrightarrow$

We have

Inclusion Loc CQ $\xrightarrow{i}$s right adjoint CQtL Loc w$\xrightarrow{j}$takes any quantale Q to $= \{ a \in Q \mid a^2 = a \}$

and homomorphism Q $\overline{Q}$ Q' to $\xrightarrow{f}$

$\overline{Q}$ $\xrightarrow{f/\overline{Q}}$ $\overline{Q}$ $'$.

2.9 An element a of a commutative quantale Q is said to be neat if it satisfies

$a \,\&\, b = a \wedge b \qquad \forall b \in Q$

Remarks

For $a, b \in Q$, a and b neat $\Rightarrow$ a & b is neat:

$\forall d \in Q$, $(a \,\&\, b) \,\&\, d = a \,\&\, (b \,\&\, d) = a \,\&\, (b \wedge d)$ (because b is neat)

$= a \wedge (b \wedge d)$ (because a is neat)

$= (a \wedge b) \wedge d$

$= (a \,\&\, b) \wedge d$

In particular $\bot$ and $1 = \top$ are neat elements of Q.

Proposition 2.6

TFAE for commutative quantale Q

i- $\forall b \in Q \qquad a \,\&\, b = a \wedge b$

ii- $x \leq a$ implies $a \,\&\, x = x$

Proof

i - $\Rightarrow$ ii $\quad x \leq a \Rightarrow a \,\&\, x = a \wedge x = x$ is clear

ii- $\Rightarrow$ i $\quad a \wedge b \leq b \Rightarrow (a \wedge b) \,\&\, a \leq b \,\&\, a$

$\Rightarrow a \wedge b \leq a \,\&\, b$, $\quad$ ($a \wedge b \leq a$ gives $(a \wedge b) \,\&\, a = a \wedge b$)

Also

$a \,\&\, b \leq a \wedge b$

because of identity of Q.

Hence $a \,\&\, b = a \wedge b$.

2.10 Let Q be a non-commutative quantale with right (left) identity, an element $a \in Q$ is said to be two sided if $a \,\&\, 1 = 1 \,\&\, a = a$

Proposition 2.7

For two sided element $a \in Q$ TFAE

i- $\forall b \in Q$, $b \,\&\, a = a \wedge b$

ii- $x \leq a$ implies $x \,\&\, a = x$

Proof

i- $\Rightarrow$ ii $x \le a \Rightarrow x \,\&\, a = x \wedge a = x$ is clear

ii- $\Rightarrow$ i Since $a \wedge b \le b$ therefore $(a \wedge b) \,\&\, a \le b \,\&\, a$

Also $a \wedge b \le a$ gives $(a \wedge b) \,\&\, a = a \wedge b$

Therefore $a \wedge b \le b \,\&\, a$.

Now $b \le 1$ gives $b \,\&\, a \le 1 \,\&\, a \le a$ (because a is two sided)

But $b \,\&\, a \le b$ therefore $b \,\&\, a \le a \wedge b$

Hence $b \,\&\, a = a \wedge b$

2.11 An element $a \in Q$ satisfying any one of above two equivalent condition is called <u>virginal</u>.

2.12 A multiple sub set $S \subseteq Q$ (that is $a, b \in S \Rightarrow a \,\&\, b \in S$) is said to be <u>basis</u> for Q if every $I \in Q$ can be written as sup of elements of S.

We define a quantale Q to be <u>coherent</u> if for basic a

$$a \le \bigvee I_i \Rightarrow a \le \bigvee_{i=1}^{n} b_i \quad (b_i \le I_i,\ b_i \text{ basic})$$

Preposition 2.8

Neat elements of a coherent quantale Q from a locale if for a, b, c, d basic,
$b \,\&\, I = b$ and $c \,\&\, J = c$ implies that there exists $d \le I \vee J$ such that $a \,\&\, d = a$ for each $a \le b \vee c$.

Proof

Suppose $k \le I \vee J$, then since Q is coherent, k can be written as join of basic elements. Suppose $a \le k$ is basic, then $a \le I \vee J$ and therefore we have $b \le I$ and

$c \le J$ such that $a \le b \vee c$, since I and J are neat $b \,\&\, I = b$ and $c \,\&\, J = c$ therefore there exists $d \le I \vee J$ such that $a \,\&\, d = a$.

Therefore

$$k \,\&\, (I \vee J) = \bigvee a_i \,\&\, (I \vee J)$$

$$= \bigvee a_i \,\&\, \bigvee d_j$$

$$= \bigvee_i \bigvee_j a_i \,\&\, d_j$$

$$= k$$

Therefore $I \vee J$ is neat.

Now for $k \le \bigvee I_i$ and basic $a \le k$ we have $a \le \bigvee I_i$ which gives

$$a \le \bigvee_{i=1}^{n} I_i$$

And therefore

$a = a \,\&\, \bigvee_{i=1}^{n} I_i$ (because I_i is neat)

$= \bigvee_{i=1}^{n} a \,\&\, I_i$

$\leq \bigvee a \,\&\, I_i$

$\leq a \,\&\, \bigvee I_i.$

Also $\quad a \,\&\, \bigvee I_i \leq a$

Therefore $\quad a \,\&\, \bigvee I_i = a$ (because each $a \,\&\, I_i \leq a$).

Hence $\quad k = \bigvee_{a_j \leq k}$

$= \bigvee_{a_j} (a_j \,\&\, \bigvee I_i)$

$= (\bigvee_{a_j} a_j) \,\&\, \bigvee I_i$

$= k \,\&\, \bigvee I_i$

Hence $\bigvee I_i$ is neat.

References

1. Johnstone, P. T. *Stone Spaces*, Cambridge studies in advanced Mathematics 3, Cambridge University Press 1977.
2. Mulvey, C.J , & , Rend : Conti Cir . Met . Palemo , 12, 99-104 (1986).
3. Mulvey C.J, Nawaz M, Quantales: Quantal Sets, Theory and Decision Library, Series B: Mathematics and Statistical Models Volum 32, Non-Classical Logics and their Applications to Fuzzy Subsets, Kluwer Academic Publishers, Netherlands , 1995.
4. Rosenthal, K.I , Quantales and their applications. Pitman Research Notes in Mathematics, series, 234, Longman Scientific and Technical Essex, U.K 1990.
5. Stout, L.N: Categories of Fuzzy Sets with values in a Quantale or Projectale, Theory and Decision Library, Series B: Mathematics.