Sixth order fixed point iterative method for solving nonlinear equations

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***Abstract*—In this paper we describe derivation of a new iterative method for solving nonlinear equations. The method is shown to have convergence order five. Furthermore we show that the convergence order may be accelerated by replacing derivative with its finite difference approximation. The accelerated convergence achieved is six. We compare our sixth order method with other methods in the literature.**

***Index Terms*—Fix Point Method, Convergence, Nonlinear Equation**

I. Introduction

OCCURRENCE of nonlinear equations contributed a critical role in applied mathematics and engineering. Therefore, over the last ten or more years, numerous mathematical methods that are intended to obtain numerical solutions of nonlinear equations arises in various fields of science and engineering. The problem of root finding of the nonlinear equations is most appropriate computational problems. This problem has widespread diversity of everyday applications in quantum mechanics, physics, chemistry, natural sciences, applied mathematics, engineering [1]. The most common root finding methods of nonlinear equations are binary search (dichotomy) method, Newton-Raphson, false?position method, secant methods, Mullers method, fix point method [1]. The binary search and false position method are the bracketing methods and Newton Raphson method, secant method, fix point method are open methods. In previous years several techniques are applied to derive new iterative methods to solve nonlinear equations. The techniques used to derive these methods includes, Taylor series, Adomian decomposition [2], Homotopy analysis method [3], and numerical quadrature [4]. In this paper we extend upon the work of Noor to derive a new method with fifth order convergence. The fifth order method involve three function evaluation where derivative is evaluated at two different points. The methods convergence can be accelerated by using finite difference which is counter intuitive. Since approximation of derivative with its finite difference approximation usually results in loss of information, while in our case accelerated convergence is obtained. The paper is organized as follows: Section two describe the construction of a third order method. Section three discusses a method to accelerate convergence, in section four numerical comparison of our method with other techniques is presented while in section five we give concluding remarks.

II. Fixed Point Method

In this section we combine three techniques namely fixed point iteration, finite difference, and interpolation to develop a new method with sixth order of convergence.Shah and Noor [5] had derived an iterative method to solve nonlinear equations. For the sake of completeness we reproduce their derivation. Let us consider the problem of solving nonlinear equation:

$$f(x) = 0 \tag{1}$$

The problem (1) can be written as fixed point iteration as follows:

$$x = F(x) \tag{2}$$

The fixed point map $F(x)$ can be chosen such as to develop an implicit or explicit iterative scheme with guaranteed convergence. Shah and Noor have considered the following $F(x)$,

$$F(x) = \phi(x) + \lambda(f(x))^p g(x) \tag{3}$$

where $\phi(x)$ is an iteration function and with convergence order p and $g(x)$ is an auxiliary function. Notice that the optimal value of λ can be obtained via optimality criteria i.e.

$$\frac{dF(x)}{dx} = 0.$$

from above the optimal value of λ come out to be

$$\lambda = -\frac{\phi'(x)}{p(f(x))^{p-1}f'(x)g(x) - (f(x))^p g'(x)}$$

Using value of λ in (3), we obtain,

$$F(x)=\phi(x)-\frac{\phi'(x)f(x)g(x)}{pf'(x)g(x)+f(x)g'(x)}.$$

Using $x_{n+1}=F(x_n)$, we have

$$x_{n+1}=\phi(x_n)-\frac{\phi'(x_n)f(x_n)g(x_n)}{pf'(x_n)g(x_n)+f(x_n)g'(x_n)} \quad (4)$$

Taking $\phi(x)=x-\frac{f(x)}{f'(x)}$ and $g(x)=e^{-\alpha x}$, the above recurrence relation reduces to,

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}-\frac{(f(x_n))^2f''(x_n)}{(f'(x_n))^2(2f'(x_n)-\alpha f(x_n))} \quad (5)$$

The above method have third order convergence but it depends upon second derivative. Shah and Noor replaced double derivative using Taylor approximation. Here instead of using Taylor approximation we can replace derivative with finite difference approximation i.e.

$$f''(x_n)\approx\frac{f'(x_n)-f'(y_n)}{x_n-y_n}$$

The above method require predictor step that is value of y_n which we can choose as follows:

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)}$$

Replacing double derivative as described above in (5) we have the following iteration:

$$x_{n+1}=y_n-\frac{1}{y_n-x_n}\frac{(f(x_n))^2(f'(y_n)-f'(x_n))}{(f'(x_n))^2(2f'(x_n)-\alpha f(x_n))} \quad (6)$$

The iteration (6) require three function evaluation and it has third order of convergence (Theorem 1) thus its efficiency index is $3^{\frac{1}{3}}=1.442$. In the next theorem we prove the order of convergence.

Theorem 1. *Let I be an open interval and $r\in I$ is a simple root of a sufficiently differentiable function f then the iteration function defined by*

$$y_n=x_n-\frac{f(x_n)}{f'(x_n)} \quad (7)$$

$$x_{n+1}=y_n-\frac{1}{y_n-x_n}\frac{(f(x_n))^2(f'(y_n)-f'(x_n))}{(f'(x_n))^2(2f'(x_n)-\alpha f(x_n))} \quad (8)$$

has convergence order three.

Proof. Let r be a simple root and since f is sufficiently differentiable thus expanding f and f' around r results in

$$f(x_n)=f'(r)\left(e_n+c_2e_n^2+c_3e_n^3+c_4e_n^4+c_5e_n^5+O(e_n^6)\right) \quad (9)$$

$$f'(x_n)=f'(r)\left(1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+5c_5e_n^4+O(e_n^5)\right) \quad (10)$$

where $c_k=\frac{1}{k!}\frac{f^{(k)}(x_n)}{f'(r)}$, $k=1,2,3,\cdots$ and $e_n=x_n-r$. The above gives:

$$y_n=r+c_2e_n^2-\left(2\,c_2^2-2\,c_3\right)e_n^3-\left(-4\,c_2^3+7\,c_2c_3-3\,c_4\right)e_n^4 -\left(8\,c_2^4-20\,{c_2}^2c_3+10\,c_2c_4+6\,c_3^2-4\,c_5\right)e_n^5+O(e_n^6)$$

Notice that,

$$f'(y_n)=f'(r)\left(1+2c_2(y_n-r)+3c_2(y_n-r)^2+4c_4(y_n-r)^3+O((y_n-r)^4)\right)$$
$$f'(y_n)=f'(r)\left(1+2\,c_2^2e_n^2+\left(-4\,c_2^3+4\,c_2c_3\right)e_n^3+\left(8\,c_2^4+3\,{c_2}^3-14\,c_2^2c_3+6\,c_2c_4\right)e_n^4+\cdots\right)$$

Using above we have,

$$(f(x_n))^2(f'(y_n)-f'(x_n))=(f'(r))^3\left(-2\,c_2e_n^3+\left(-2\,c_2^2-3\,c_3\right)e_n^4+\left(-2\,c_2^3-6\,c_2c_3-4\,c_4\right)e_n^5\right)$$
$$(f'(x_n))^2(2f'(x_n)-\alpha f(x_n))=(f'(r))^3(2+(-\alpha+12\,c_2)\,e_n+\left(-5\,\alpha\,c_2+24\,c_2^2+18\,c_3\right)e_n^2$$
$$+\left(-8\,\alpha\,c_2^2+16\,c_2^3-7\,\alpha\,c_3+72\,c_2c_3+24\,c_4\right)e_n^3$$
$$+\left(-4\,\alpha\,{c_2}^3-22\,\alpha\,c_2c_3+72\,c_2^2c_3-9\,\alpha\,c_4+96\,c_2c_4+54\,c_3^2+30\,c_5\right)e_n^4+\cdots$$

Thus we have,

$$e_{n+1}=\left(1/2\,c_3+2\,c_2^2-1/2\,\alpha\,c_2\right)e_n^3+\left(c_4-9\,c_2^3-1/4\,\alpha^2c_2+9/2\,c_2c_3+5/2\,\alpha\,c_2^2-3/4\,\alpha\,c_3\right)e_n^4+\cdots$$

III. Accelerated Convergence

It is well known that by combining method with other methods as predictor corrector scheme their convergence order can be improved. Thus to accelerate the convergence of our method we combine it with another iteration of Newton's method as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - \frac{1}{y_n - x_n} \frac{(f(x_n))^2 (f'(y_n) - f'(x_n))}{(f'(x_n))^2 (2f'(x_n) - \alpha f(x_n))} \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)} \end{aligned} \tag{11}$$

The order of convergence for above method is six which can be easily verified. The efficiency index for the above method is $6^{\frac{1}{5}} = 1.43$, which is actually reduced. To increase the efficiency index we ought to reduce the number of function evaluation. To decrease the number of function evaluation we approximate derivative at z_n using linear interpolation of two points namely $(x_n, f'(x_n))$ and $(y_n, f'(y_n))$ i.e.

$$f'(x) \approx \frac{x - x_n}{y_n - x_n} f'(y_n) + \frac{x - y_n}{x_n - y_n} f'(x_n)$$

By using the above we have,

$$f'(z_n) \approx \frac{f'(x_n)(f'(x_n) - 4f'(y_n)) + f'(y_n)(\alpha f(x_n) + f'(x_n)))}{\alpha f(x_n) - 2f'(x_n)}$$

Notice that by replacing $f'(z_n)$ with its approximation does not effect the order of convergence.

Theorem 2. *Let I be an open interval and $r \in I$ is a simple root of a sufficiently differentiable function f then the iteration function defined by*

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} \tag{12}$$

$$z_n = y_n - \frac{1}{y_n - x_n} \frac{(f(x_n))^2 (f'(y_n) - f'(x_n))}{(f'(x_n))^2 (2f'(x_n) - \alpha f(x_n))} \tag{13}$$

$$x_{n+1} = z_n - \frac{f(z_n)(\alpha f(x_n) - 2f'(x_n))}{f'(x_n)(f'(x_n) - 4f'(y_n)) + f'(y_n)(\alpha f(x_n) + f'(x_n)))} \tag{14}$$

has convergence order six.

Proof. In lieu of Theorem 1, we have,

$$\begin{aligned} z_n = {} & r + \left(2\,{c_2}^2 + \frac{1}{2}\,c_3 - \frac{1}{2}\,\alpha\,c_2\right){e_n}^3 + \left(-9\,{c_2}^3 + c_4 - \frac{1}{4}\,\alpha^2 c_2 + \frac{5}{2}\,\alpha\,{c_2}^2 - \frac{3}{4}\,\alpha\,c_3 + \frac{9}{2}\,c_2 c_3\right){e_n}^4 \\ & + \left(\begin{array}{l} -\frac{1}{8}\,\alpha^3 c_2 + \left(\frac{3}{2}\,{c_2}^2 - 3/8\,c_3\right)\alpha^2 + \left(\frac{15}{2}\,c_2 c_3 - \frac{19}{2}\,{c_2}^3 - c_4\right)\alpha \\ +30\,{c_2}^4 + \frac{3}{2}\,{c_2}^3 + \frac{3}{2}\,{c_3}^2 + \frac{3}{2}\,c_5 - 36\,{c_2}^2 c_3 + 6\,c_2 c_4 \end{array}\right){e_n}^5 \\ & + \left(\begin{array}{l} 2\,c_6 - 88\,{c_2}^5 + \frac{15}{2}\,c_2 c_5 - \frac{21}{2}\,{c_2}^4 - \frac{5}{4}\,\alpha\,c_5 - \frac{13}{2}\,\alpha^2 {c_2}^3 + \frac{63\,\alpha\,{c_2}^4}{2} - \frac{1}{2}\,\alpha^2 c_4 + \frac{7\,\alpha^3 {c_2}^2}{8} \\ -\frac{3}{16}\,\alpha^3 c_3 - \frac{1}{16}\,\alpha^4 c_2 + \frac{3}{4}\,\alpha\,{c_2}^3 + \frac{21\,\alpha\,{c_3}^2}{4} - \frac{171\,\alpha\,{c_2}^2 c_3}{4} + \frac{21}{2}\,\alpha\,c_2 c_4 + \frac{37\,\alpha^2 c_2 c_3}{8} \\ -50\,{c_2}^2 c_4 + \frac{7}{2}\,c_3 c_4 + 167\,{c_2}^3 c_3 - 43\,c_2 {c_3}^2 + 6\,{c_2}^2 c_3 \end{array}\right){e_n}^6 \\ & + \left(\begin{array}{l} \frac{5}{2}\,c_7 + 48\,{c_2}^5 + 2\,{c_4}^2 - \frac{27\,{c_3}^3}{2} + 240\,{c_2}^6 + 9\,c_2 c_6 - 118\,c_2 c_3 c_4 + \frac{27\,\alpha^2 {c_3}^2}{8} - \frac{5}{8}\,\alpha^2 c_5 \\ -\frac{17\,\alpha^3 {c_2}^3}{4} + 24\,\alpha^2 {c_2}^4 - 96\,\alpha\,{c_2}^5 - \frac{1}{4}\,\alpha^3 c_4 + \frac{1}{2}\,\alpha^4 {c_2}^2 - \frac{3\,\alpha^4 c_3}{32} - \frac{1}{32}\,\alpha^5 c_2 + \frac{3}{8}\,\alpha^2 {c_2}^3 \\ -6\,\alpha\,{c_2}^4 - \frac{3}{2}\,\alpha\,c_6 + \frac{11}{4}\,\alpha^3 c_2 c_3 + 3\,\alpha\,{c_2}^2 c_3 - 60\,\alpha\,{c_2}^2 c_4 - \frac{123\,\alpha\,c_2 {c_3}^2}{2} + \frac{27\,\alpha\,c_2 c_5}{2} \\ +\frac{29\,\alpha\,c_3 c_4}{2} - \frac{241\,\alpha^2 {c_2}^2 c_3}{8} + 189\,\alpha\,{c_2}^3 c_3 + \frac{13}{2}\,\alpha^2 c_2 c_4 + 9\,{c_2}^2 c_4 - \frac{117\,{c_2}^3 c_3}{2} + 6\,c_2 {c_3}^2 \\ -64\,{c_2}^2 c_5 + 4\,c_3 c_5 + 234\,{c_2}^3 c_4 - 624\,{c_2}^4 c_3 + 330\,{c_2}^2 {c_3}^2 \end{array}\right){e_n}^7 \\ & + \left(\begin{array}{l} 3\,c_8 - 180\,{c_2}^6 - 624\,{c_2}^7 - 150\,c_2 c_3 c_5 + 924\,{c_2}^2 c_3 c_4 + 18\,c_2 c_3 c_4 + \frac{21}{2}\,c_2 c_7 + \\ \frac{9\,\alpha^5 {c_2}^2}{32} - \frac{3\,\alpha^5 c_3}{64} - \frac{\alpha^6 c_2}{64} - \frac{111\,\alpha\,{c_3}^3}{4} + 10\,\alpha\,{c_4}^2 - \frac{7}{4}\,\alpha\,c_7 - \frac{3}{4}\,\alpha^2 c_6 + \frac{33\,\alpha^3 {c_3}^2}{16} \\ -\frac{5\,\alpha^3 c_5}{16} - \frac{43\,\alpha^4 {c_2}^3}{16} + \frac{69\,\alpha^3 {c_2}^4}{4} - \frac{321\,\alpha^2 {c_2}^5}{4} + 276\,\alpha\,{c_2}^6 - \frac{1}{8}\,\alpha^4 c_4 + \frac{3}{16}\,\alpha^3 {c_2}^3 \\ -\frac{27\,\alpha^2 {c_2}^4}{8} + \frac{123\,\alpha\,{c_2}^5}{4} - \frac{309\,\alpha\,{c_2}^2 c_5}{4} + \frac{33\,\alpha\,c_2 c_6}{2} + \frac{37\,\alpha\,c_3 c_5}{2} - \frac{85\,\alpha^2 {c_2}^2 c_4}{2} \\ -45\,\alpha^2 c_2 {c_3}^2 + \frac{531\,\alpha\,{c_2}^3 c_4}{2} + \frac{1653\,\alpha\,{c_2}^2 {c_3}^2}{4} + \frac{67\,\alpha^2 c_2 c_5}{8} + \frac{75\,\alpha^2 c_3 c_4}{8} - \frac{161\,\alpha^3 {c_2}^2 c_3}{8} \\ +\frac{1185\,\alpha^2 {c_2}^3 c_3}{8} - 720\,\alpha\,{c_2}^4 c_3 + \frac{31\,\alpha^3 c_2 c_4}{8} + \frac{51\,\alpha^4 c_2 c_3}{32} + \frac{9}{2}\,\alpha\,{c_2}^2 c_4 + 3\,\alpha\,c_2 {c_3}^2 + \frac{3}{2}\,\alpha^2 {c_2}^2 c_3 \\ -\frac{135\,\alpha\,{c_2}^3 c_3}{4} + 12\,{c_2}^2 c_5 - \frac{171\,{c_2}^3 c_4}{2} + 342\,{c_2}^4 c_3 - 108\,{c_2}^2 {c_3}^2 - 78\,{c_2}^2 c_6 + \frac{9}{2}\,c_3 c_6 + 297\,{c_2}^3 c_5 + \\ \frac{9}{2}\,c_4 c_5 - 882\,{c_2}^4 c_4 - 81\,c_2 {c_4}^2 - 54\,{c_3}^2 c_4 + 2064\,{c_2}^5 c_3 - 1710\,{c_2}^3 {c_3}^2 + \frac{531\,c_2 {c_3}^3}{2} - \frac{343\,\alpha\,c_2 c_3 c_4}{2} \end{array}\right){e_n}^8 + \cdots \end{aligned}$$

We have,

$$f(z_n) = f'(r)(z_n - r + c_2 * (z_n - r)^2 + c_3 * (z_n - r)^3 + c_4 * (z_n - r)^4) + \cdots$$

Thus giving,

$$\begin{aligned}
f(z_n) =& f'(r)\left(2\,{c_2}^2 + 1/2\,c_3 - 1/2\,\alpha\,c_2\right){e_n}^3 + f'(r)\left(-9\,{c_2}^3 + c_4 - 1/4\,\alpha^2 c_2 + 5/2\,\alpha\,{c_2}^2 - 3/4\,\alpha\,c_3 + 9/2\,c_2 c_3\right){e_n}^4 \\
&+ f'(r)\begin{pmatrix} 30\,{c_2}^4 + 3/2\,{c_2}^3 + 3/2\,{c_3}^2 + 3/2\,c_5 + \frac{15}{2}\,\alpha\,c_2 c_3 - 1/8\,\alpha^3 c_2 \\ + 3/2\,\alpha^2 {c_2}^2 - 19/2\,\alpha\,{c_2}^3 - 3/8\,\alpha^2 c_3 - 36\,{c_2}^2 c_3 - \alpha\,c_4 + 6\,c_2 c_4 \end{pmatrix}{e_n}^5 \\
&+ f'(r)\begin{pmatrix} 2\,c_6 + 15/2\,c_2 c_5 - 5/4\,\alpha\,c_5 - \frac{25\,\alpha^2 {c_2}^3}{4} + \frac{59\,\alpha\,{c_2}^4}{2} - 1/2\,\alpha^2 c_4 + \frac{7\,\alpha^3 {c_2}^2}{8} \\ - 3/16\,\alpha^3 c_3 - 1/16\,\alpha^4 c_2 + 3/4\,\alpha\,{c_2}^3 + \frac{21\,\alpha\,{c_3}^2}{4} - 50\,{c_2}^2 c_4 + 7/2\,c_3 c_4 + 169\,{c_2}^3 c_3 \\ - \frac{171\,c_2 {c_3}^2}{4} + 6\,{c_2}^2 c_3 - 84\,{c_2}^5 - 21/2\,{c_2}^4 - \frac{173\,\alpha\,{c_2}^2 c_3}{4} + 21/2\,\alpha\,c_2 c_4 + \frac{37\,\alpha^2 c_2 c_3}{8} \end{pmatrix}{e_n}^6 \\
&+ f'(r)\begin{pmatrix} \frac{5}{2}\,c_7 + 48\,{c_2}^5 + 2\,{c_4}^2 - \frac{27\,{c_3}^3}{2} + 204\,{c_2}^6 + 9\,c_2 c_6 - 117\,c_2 c_3 c_4 + \frac{27\,\alpha^2 {c_3}^2}{8} - \frac{5}{8}\,\alpha^2 c_5 \\ - 4\,\alpha^3 {c_2}^3 + \frac{41\,\alpha^2 {c_2}^4}{2} - 77\,\alpha\,{c_2}^5 - \frac{1}{4}\,\alpha^3 c_4 + \frac{1}{2}\,\alpha^4 {c_2}^2 - \frac{3\,\alpha^4 c_3}{32} - \frac{1}{32}\,\alpha^5 c_2 + \frac{3}{8}\,\alpha^2 {c_2}^3 \\ - 6\,\alpha\,{c_2}^4 - \frac{3}{2}\,\alpha\,c_6 + \frac{11}{4}\,\alpha^3 c_2 c_3 + 3\,\alpha\,{c_2}^2 c_3 - 61\,\alpha\,{c_2}^2 c_4 - \frac{249\,\alpha\,c_2 {c_3}^2}{4} + \frac{27\,\alpha\,c_2 c_5}{2} \\ + \frac{29\,\alpha\,c_3 c_4}{2} - \frac{237\,\alpha^2 {c_2}^2 c_3}{8} + 184\,\alpha\,{c_2}^3 c_3 + \frac{13}{2}\,\alpha^2 c_2 c_4 + 9\,{c_2}^2 c_4 \\ - \frac{117\,{c_2}^3 c_3}{2} + 6\,c_2 {c_3}^2 - 64\,{c_2}^2 c_5 + 4\,c_3 c_5 + 238\,{c_2}^3 c_4 - 615\,{c_2}^4 c_3 + \frac{669\,{c_2}^2 {c_3}^2}{2} \end{pmatrix}{e_n}^7 \\
&+ f'(r)\begin{pmatrix} 3\,c_8 - 174\,{c_2}^6 - 423\,{c_2}^7 - \frac{297\,c_2 c_3 c_5}{2} + 939\,{c_2}^2 c_3 c_4 + 18\,c_2 c_3 c_4 + \frac{21}{2}\,c_2 c_7 + \frac{9\,\alpha^5 {c_2}^2}{32} \\ - \frac{3\,\alpha^5 c_3}{64} - \frac{\alpha^6 c_2}{64} - \frac{111\,\alpha\,{c_3}^3}{4} + 10\,\alpha\,{c_4}^2 - \frac{7}{4}\,\alpha\,c_7 - \frac{3}{4}\,\alpha^2 c_6 + \frac{33\,\alpha^3 {c_3}^2}{16} - \frac{5\,\alpha^3 c_5}{16} \\ - \frac{5}{2}\,\alpha^4 {c_2}^3 + 14\,\alpha^3 {c_2}^4 - 54\,\alpha^2 {c_2}^5 + 163\,\alpha\,{c_2}^6 - \frac{1}{8}\,\alpha^4 c_4 + \frac{3}{16}\,\alpha^3 {c_2}^3 - \frac{27\,\alpha^2 {c_2}^4}{8} \\ + \frac{117\,\alpha\,{c_2}^5}{4} - \frac{315\,\alpha\,{c_2}^2 c_5}{4} + \frac{33\,\alpha\,c_2 c_6}{2} + \frac{37\,\alpha\,c_3 c_5}{2} - 42\,\alpha^2 {c_2}^2 c_4 - \frac{717\,\alpha^2 c_2 {c_3}^2}{16} \\ + \frac{521\,\alpha\,{c_2}^3 c_4}{2} + \frac{825\,\alpha\,{c_2}^2 {c_3}^2}{2} + \frac{67\,\alpha^2 c_2 c_5}{8} + \frac{75\,\alpha^2 c_3 c_4}{8} - \frac{39\,\alpha^3 {c_2}^2 c_3}{2} + \frac{1077\,\alpha^2 {c_2}^3 c_3}{8} \\ - \frac{1255\,\alpha\,{c_2}^4 c_3}{2} + \frac{31\,\alpha^3 c_2 c_4}{8} + \frac{51\,\alpha^4 c_2 c_3}{32} + \frac{9}{2}\,\alpha\,{c_2}^2 c_4 + 3\,\alpha\,c_2 {c_3}^2 + \frac{3}{2}\,\alpha^2 {c_2}^2 c_3 \\ - \frac{135\,\alpha\,{c_2}^3 c_3}{4} + 12\,{c_2}^2 c_5 - \frac{171\,{c_2}^3 c_4}{2} + \frac{687\,{c_2}^4 c_3}{2} - 108\,{c_2}^2 {c_3}^2 - 78\,{c_2}^2 c_6 + \frac{9}{2}\,c_3 c_6 \\ + 303\,{c_2}^3 c_5 + \frac{9}{2}\,c_4 c_5 - 876\,{c_2}^4 c_4 - 80\,c_2 {c_4}^2 - 54\,{c_3}^2 c_4 + 1869\,{c_2}^5 c_3 \\ - \frac{6879\,{c_2}^3 {c_3}^2}{4} + 267\,c_2 {c_3}^3 - 174\,\alpha\,c_2 c_3 c_4 \end{pmatrix}{e_n}^8 \\
&+ \cdots
\end{aligned}$$

Thus simplifying the above relation we get:

$$e_{n+1} = \left(1/4\,\alpha^2 {c_2}^3 - 2\,\alpha\,{c_2}^4 + 4\,{c_2}^5 - 4\,{c_2}^3 c_3 - 5/4\,c_2 {c_3}^2 + \alpha\,{c_2}^2 c_3\right){e_n}^6 + \cdots$$

Thus the method had sixth order convergence.

Note that with the change of interpolating point with $f'(z_n)$ the method retain its convergence order and its efficiency index is improved which is now 1.56. Here it is worth mentioning that changing derivative with linear interpolation polynomial is well-known see e.g. [6] and references therein.

IV. Numerical Results

In this section we present comparison of our method with other six order method. The method are sixth order modification of well known techniques. The methods are implemented on Maple 15 with 300 digits precision. We used absolute error between consecutive iterations as a stopping criteria with tolerance 10^{-50} i.e $|x_{n+1} - x_n| < 10^{-50}$. We have also calculated computational order of convergence which is given below:

$$COC = \frac{ln\left|\frac{x_{n+1}-x_n}{x_n-x_{n-1}}\right|}{ln\left|\frac{x_n-x_{n-1}}{x_{n-1}-x_{n-2}}\right|}.$$

For our method we choose $\alpha = 0.5$. The results are given in Table I.

V. Conclusion and Discussion

It is clear from Table I that our proposed method works well in comparison with other methods. In the worst case the method performs as good as the other methods. In this paper we have made calculations by fixing α, in future we would like to analyze the dependence of our method to α. Furthermore we analysis the stability of the method for the complex roots.

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TABLE I
NUMERICAL COMPARISON OF ITERATIVE METHODS. NOTE THAT NUMBER OF ITERATION TO FIND ZERO ALONG WITH COMPUTATIONAL ORDER OF CONVERGENCE IS REPORTED. R METHOD [7], W METHOD[8], P METHOD[9], G METHOD[10].

Equation	x_0		New Method	R Method	W Method	P Method	G Method
$x - 2 - e^{-x}$	3	It.	4	5	7	4	4
		CoC	5.9731	2.8746	1.9719	5.9575	5.9584
$x^2 - e^x - 3x + 2$	0	It.	4	5	7	4	4
		CoC	5.9959	2.9877	2.0076	5.9993	5.9839
$xe^{x^2} - sin^2(x) + 3cos(x) + 5$	-1	It	4	Div	9	4	4
		CoC	6.5515	0.0000	2.4249	6.0225	5.9708
$sin(x)e^x + ln(x^2 + 1)$	2	It.	5	7	8	18	5
		CoC	4.6704	2.8044	1.7698	1.7957	4.9924
$(x - 1)^3 - 1$	3	It.	4	10	50	4	4
		CoC	5.4610	5.4153	-7.1612	5.7183	5.5318
$sin^2(x) - x^2 + 1$	1	It.	5	11	10	4	4
		CoC	6.7395	0.8440	1.9722	6.1312	6.1252
$x^2 - 0.03x + 0.0002$	1.5	It.	7	10	18	7	7
		CoC	3.2351	1.9310	0.5717	3.4535	3.2286
$x^2 - (1 - x)^5$	0.2	It.	5	9	10	5	4
		CoC	5.2484	2.4857	2.0209	5.2198	5.5429
$e^{x^2+7x-30} - 1$	3.5	It.	6	12	8	6	6
		CoC	9.1874	2.2970	1.7734	5.3355	11.0127
$sin(x) - \frac{1}{2}x$	2	It.	4	5	7	4	4
		CoC	5.9787	2.9837	1.9722	5.9854	5.9857