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Research Article

Quadrature Rule Based Iterative Method for the Solution of Non-Linear Equations

Umair Khalid Qureshi[1*], Irfan Ali Bozdar[2], Aliya Pirzada[3], and Muzaffar Bashir Arain[4]

[1]Department of Business Administration, Shaheed Benazir Bhutto University, Sanghar, Sindh
[2]Department of Information Technology, Shaheed Benazir Bhutto University, Shaheed Benaziraba, Sindh
[3]Department of Business Administration, Shaheed Benazir Bhutto University, Naushahro Feroze, Sindh
[4]Department of Mathematics and Statistics, Quaid-e-Awam University of Engineering, Science & Technology, Nawabshah, Sindh

Abstract: This research has suggested a quadrature rule based iterative method for the solution of non-linear algebraic and transcendental equations. The proposed iterated method is derived from Quadrature Formula and Numerical Technique. The quadrature rule based iterative method is converged quadratically, and it is free from pitfall. Few of physical non-linear problems to demonstrate the competency of proposed iterative method with the assessment of Steffensen Method and Newton Raphson Method. C++ and EXCEL have been used to examine the numerical results and graphical illustration of quadrature rule based iterative method. Hence, from several examples illustrate that the convergence and efficiency of the quadrature rule based iterative method is better than Steffensen Method and Newton Raphson Method.

Keywords: Nonlinear problems, Second order method, Quadrature formula, Convergence analysis, Error.

1. INTRODUCTION

For estimating nonlinear equations are most important and useful problems, which arises in a varied collection of practical applications in engineering and applied sciences [1] for example: Distance, rate, time problems, population change, Trajectory of a ball, etc, such as non-linear equation

$$f(x) = 0 \tag{1.1}$$

Numerous techniques had been investigated for the importance of Eq. (1.1), such as one of the most effective technique is Quadrature rule, which is most useful and vigorous numerical techniques for solving numerical integration. Quadrature rule is not only useful in numerical integration but most of the researcher have been developed a technique with the help of Quadrature rule for solving the roots of nonlinear equations [2]. Recently several modifications had been done by using Quadrature rule for solving nonlinear equations [3-5]. Furthermore, increasing order of convergence and computational efficiency become an efficient way to obtain approximate solution and it has been vigorous field of research. So, various numerical methods have been developed for order of convergence and computational efficiency by using quadrature formula with the combination of different techniques including finite differences, Taylor series and decomposition method [6-11]. Similarly, this study is presented a numerical iterated method which is the combination of Quadrature rule and numerical techniques. Few problems have exemplified to the performance of Quadrature Rule Based Iterative Method.

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* Corresponding Author: Umair Khalid Qureshi <umair.khalid_sng@sbbusba.edu.pk>

2. QUADRATURE METHOD

This segment developed a quadrature rule based iterated method for solving nonlinear equations with the help of Quadrature Rule and Numerical Technique. Let Quadrature Rule

$$\int_{x_0}^{x} f`(x)dx = \frac{h}{2}[f`(x_0) + 2f`(x_1) + f`(x_2)]$$

The placement of certain integration, we get the following statement,

$$f(x) = f(x_0) + \frac{h}{2}[f`(x_0) + 2f`(x_1) + f`(x_2)] \quad (2.1)$$

Where n=2 then h become,

$$h = \frac{x - x_0}{2}$$

h substitutes in Eq. (2.1), then

$$f(x) = f(x_0) + (x - x_0)\left[\frac{f`(x_0) + 2f`(x_1) + f`(x_2)}{4}\right] \quad (2.2)$$

Where is an initial guess, which is near to root `x` and using Eq. (1.1), then Eq. (2.2) become as

$$x = x_0 - \frac{4f(x_0)}{f`(x_0) + 2f`(x_1) + f`(x_2)} \quad (2.3)$$

Now using numerical technique, such as

$$x_1 = x_0 + h$$

or

$$x_2 = x_0 + 2h$$

To using $h = f(x^0)$ from reference [12], such as

$$x_1 = x_0 + f(x_0) \quad (2.4)$$

or

$$x_2 = x_0 + 2f(x_0) \quad (2.5)$$

Substitute Eq. (2.4) and Eq. (2.5) in Eq. (2.3), we get

$$x = x_0 - \frac{4f(x_0)}{f`(x_0) + 2f`(x_0 + f(x_0)) + f`(x_0 + 2f(x_0))}$$

In general

$$x_{n+1} = x_n - \frac{4f(x_n)}{f`(x_n) + 2f`(x_n + f(x_n)) + f`(x_n + 2f(x_n))} \quad (2.6)$$

Hence, Eq. (2.6) is a Quadrature Rule Based Iterated Method for Solving Nonlinear Problems.

3. CONVERGENCE ANALYSIS

This section is giving the key results of this paper. We have given here the Mathematical proof that the proposed Method is converge quadratically.

Proof:

By using Taylor series, we are expanding, $f(x_n)$, $f`(x_n)$, $f`(x_n+f(x_n))$ and $f`(x_n+2f(x_n))$ only second order term about `a`, such as

$$f(x_n) = e_n f`(a)(1 + ce_n) \quad (3.1)$$

$$f`(x_n) = f`(a)(1 + 2ce_n) \quad (3.2)$$

and

$$f(x_n + f(x_n)) = f`(a)\left[(e_n + f(x_n)) + (e_n + f(x_n))^2 c\right]$$

Or

$$f'(x_n + f(x_n)) = f`(a)[(1 + f'(x_n)) + 2(e_n + f(x_n))(1 + f'(x_n))c]$$

Eq. and Eq. substitute in above Equation, we get

$$f`(x_n + f(x_n)) = f`(a)[1 + f`(a) + 2ce_n + 6ce_n f`(a) + 2ce_n f`^2(a)] \quad (3.3)$$

Now,

$$f(x_n + 2f(x_n)) = f`(a)[(e_n + 2f(x_n)) + c(e_n + 2f(x_n))^2]$$

or

$$f`(x_n + 2f(x_n)) = f`(a)[(1 + 2f`(x_n)) + 2c(e_n + 2f(x_n))(1 + 2f`(x_n))]$$

Using Eq. and Eq. in above, we get

$$f`(x_n + 2f(x_n)) = f`(a)[1 + 2f`(a) + 2ce_n + 12ce_n f`(a) + 8ce_n f`^2(a)] \quad (3.4)$$

For solving Eq. , Eq. (3.3) and Eq. (3.4), we get

$$f`(x_n) + 2f`(x_n + f(x_n)) + f`(x_n + 2f(x_n))$$
$$= 4f`(a)[1 + f`(a) + ce_n(2 + 6f`(a) + 3f`^2(a))] \quad (3.5)$$

Now using Eq. (3.1) and Eq. (3.5) in developed method, we get

$$e_{n+1} = e_n - \frac{4e_n f`(a)(1 + ce_n)}{4f`(a)[1 + f`(a) + ce_n(2 + 6f`(a) + 3f`^2(a))]}$$

$$e_{n+1}=-e_n f`(a)-ce^2{}_n\left(3-5f`(a)-3f`^2(a)\right) \quad (3.6)$$

Finally, Eq. (1.1) using in Eq. (3.1) then put in Eq. (3.6), we get

$$e_{n+1} = f``(a)e_n{}^2 - ce^2{}_n\left(3 - 5f`(a) - 3f`^2(a)\right)$$

or

$$e_{n+1}=e^2{}_n[f``(a)-c\left(3-5f`(a)-3f`^2(a)\right)] \quad (3.7)$$

Hence, Eq. (3.7) has been proven that the proposed method is converge quadratically.

4. RESULTS AND DISCUSSION

The proposed quadrature rule based iterated method is applied on few physical non-linear problems. The developed method equated with the Steffensen Method and Newton Raphson Method, physical problems are:

Example-1
The equation $sin^2x - x^2+1=0$ governing the mass of the jumper. Use the suitable techniques to compute approximately the mass of the jumper under free fall.

Example-2
Find the root of $f(x) = 0$. The pollutant bacteria concentration in a lake varies as $2x - lnx - 7 = 0$ with $x_0 = 6$, then calculate the displacement `x` for the bacteria concentration.

Example-3
The volume of the gas depends on the temperatures. Volume v_1 and v_2 of two gases under ideal situation given by $v_1= e^{-x}$ and $v_2=cosx$`x` being the temperature. For what value of `x` are the volumes of the gases equal (such as $e^{-x} - cosx=0$ with $x_0=4$).

Example-4
Find the diameter `x` of the pipe which satisfies the flow equation $2x^2 - 5x - 2=0$ with $x_0=0$.
Example-5
In the analysis of the anti-symmetric buckling of a beam, a factor `x` satisfies (0, 2) and $x^2 - e^x$. Determine the `x` by using any techniques.

For the assessment of these physical problems which are present in Table 1 as well as graphical representation in Figure 1, Figure 2 & Figure 3). It has been perceived that the developed technique is reducing the number of iterations and decent accuracy viewpoint as the comparison of Steffensen Method and Newton Raphson Method.

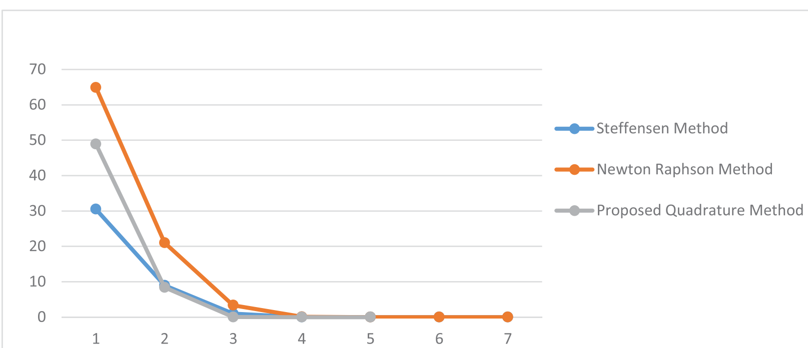

Fig. 1. Comparative Illustration A.E% of 'mass of the jumper' that is $\sin^2x\text{-}x^2+1=0$

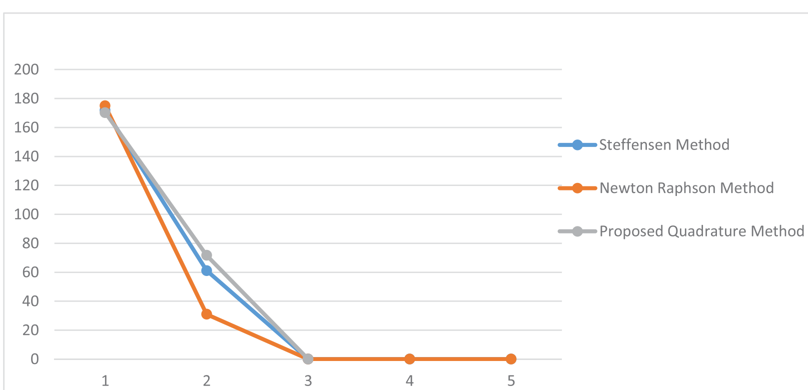

Fig. 2. Comparative Illustration A.E% 'Bacteria Concentration' that is 2x-lnx-7=0

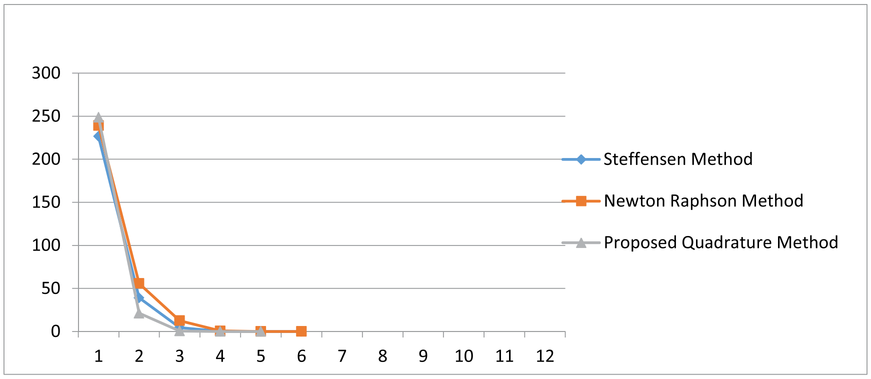

Fig. 3. Comparative Illustration A.E% of 'volumes of the gas' that is x^2--e^x=0

Table 1. Assessment of physical problems

Functions	Initial guess	Methods	Iterations	Root	A E %
$\sin^2 x - x^2 + 1 = 0$	$x_0 = 1$	Steffensen Method	7		2.22045e-016
		Newton Raphson Method	7	1.40449	2.22045e-016
		Proposed Quadrature Method	5		2.22045e-016
$2x - \ln x - 7 = 0$	$x_0 = 6$	Steffensen Method	5		8.88178e-016
		Newton Raphson Method	5	4.219906	8.88178e-016
		Proposed Quadrature Method	3		3.97682e-004
$x^2 - e^x = 0$	$x_0 = 2$	Steffensen Method	6		1.42109e-014
		Newton Raphson Method	6	0.703467	1.01070e-010
		Proposed Quadrature Method	5		1.18039e-012
$2x^2 - 5x - 2 = 0$	$x_0 = 0$	Steffensen Method	6		2.40680e-012
		Newton Raphson Method	6	0.35079	5.55112e-017
		Proposed Quadrature Method	5		1.88228e-011
$e^{-x} - \cos x = 0$	$x_0 = 4$	Steffensen Method	3		1.64524e-009
		Newton Raphson Method	4	4.72129	1.23767e-008
		Proposed Quadrature Method	3		4.95920e-009

5. CONCLUSION

In this study a quadrature rule based iterated method has been constructed for solving application non-linear problems. The proposed iterated method is based on quadrature rule and numerical technique. Proposed method proven that the develop algorithm is converging quadratically. During the study, it has been concluded that the proposed method is loftier than Steffensen Method and Newton Raphson Method from accuracy outlook as well as iterative perception. Henceforth the quadrature rule based iterated method is converged rapidly, more competent and perform-well as the assessment of Steffensen Method and Newton Raphson Method for solving the non-linear application problems.

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