Proceedings of the Pakistan Academy of Sciences:
A. Physical and Computational Sciences 56 (1): 69–75 (2019)

ISSN: 2518-4245 (print), 2518-4253 (online)

Pakistan Academy of Sciences

Research Article

A Comparison of Modality Tests Based on Real Life Data Applications

Rana Muhammad Imran Arshad[1], Shahina Imran[2], Farrukh Jamal[3*], and Gamze Ozel[4]

[1]Department of Statistics Govt., S.E. College, Bahawalpur, Pakistan
[2]Department of Economics Govt., S. E. College, Bahawalpur, Pakistan
[3]Department of Statistics Govt., S.A. Postgraduate College, Dera Nawab Sahib, Bahawalpur, Pakistan
[4]Department of Statistics, Hacettepe University, Ankara, Turkey

Abstract: Nonparametric tests for the modality are known as distribution-free tests to evaluate the evidence about homogeneity in a population. Silverman's bandwidth test by Silverman [1], the Hartigan's dip test by Hartigan [6], Hartigan and Hartigan [7], Proportional Mass (PM) test by Cavallo and Ringobon [4] and Excess Mass (EM) test by Muller and Sawitzki [9] are very popular modality tests in literature. The researchers of these tests claimed that their tests perform well to detection of unimodality, bimodality and multimodality. This study focusses on the comparison of these tests based on real data applications. The results show that Silverman's bandwidth test is the best test for detecting modality whether it is unimodality, bimodality or multimodality.

Keywords: Nonparametric tests, Silverman's bandwidth test, Hartigan's dip test, Proportional Mass (PM) test, Excess Mass test, unimodality, Real life data application.

1. INTRODUCTION

Modality tests are important for real life data examples where the distribution is bimodal and multimodal. These tests can be used in economics, social sciences, sports sciences, medical sciences, biological sciences, etc. In this study, we conduct four famous non-parametric tests for modality on several real-life data applications. These four tests are Silverman's bandwidth test by Silverman [1], the Hartigan's dip test proposed by Hartigan [6], Hartigan and Hartigan [7], proportional mass (PM) test of Cavallo and Ringobon [4] and excess mass (EM) test of Muller and Sawitzki [9]. Silverman's bandwidth test is a famous analysis for assessing whether the univariate distribution has k modes [10, 11]. The main advantage of this test comes from its computational simplicity since it is useful to test multiple hypotheses. Silverman [10] obtained that the Kernel density estimate is unimodal employing the smallest window width as a test statistic for unimodality. The significance level of the test statistic is obtained by sampling from a rescaled form of the unimodal density estimate.

Hartigan's dip test is used to show the dip statistic to test for unimodality. It is calculated by the maximum difference between the empirical distribution function and the unimodal distribution function which minimizes the maximum difference. Hence, the dip measures proposed by Hartigan and Hartigan [7] is as a test statistic for unimodality. Hartigan [6] obtained a Fortran code and then Maechler provided corrected C code as part of an R package dip test. The disadvantage of the Hartigan's and Silverman's tests is that the null hypothesis is strict unimodality. Therefore, it leads to rejection. For this limitation, the PM test was developed by Cavallo and Ringobon [4]. This test is used to find unimodality around specific value

Received: February 2018; Accepted: March 2019
* Corresponding Author: Farrukh Jamal <farrukhjamalmphil@gmail.com>

for the distribution and to allow for small modes in the distribution as part of the null-hypothesis [5]. Müller and Sawitzki [9] proposed the EM test for the statistical inference of multi-modality. It is a general method for statistical analysis and can be used to obtain a probabilistic model for clustering. The EM test can also be applied for the analysis of multi-modality. Hence, a mode is present where an excess of the probability mass is concentrated. This idea can be formalized directly using the EM functional [5]. The EM measures the local difference of a given distribution to a reference model which is generally the uniform distribution.

There are many studies based on the application of modality tests. Cheng and Hall [3] developed a practical approach for calibrating the excess mass and dip tests to improve their level accuracy and power substantially. Bianchi [2] used the convergence hypothesis in a cross-section of 119 countries from bootstrap multimodality tests and nonparametric density estimation techniques. Xu, Ling, Bedrick, E., Hanson, T., Restrepo, C. compared the performance of Hartigan's dip, Silverman's bandwidth, PM and EM tests statistical methods to test for departures from unimodality in simulations, and further illustrate the four methods using well-known ecological datasets on body mass to illustrate their advantages and disadvantages. Cavallo and Rigobon [4] compared the results of Hartigan's dip, Silverman's bandwidth and the PM tests based on prices change data. These four methods using well-known ecological datasets on body mass to illustrate their advantages and disadvantages. Cavallo and Rigobon [4] compared the results of Hartigan's dip, Silverman's bandwidth and the PM tests based on prices change data.

The motivation of this study is to compare Hartigan's dip, Silverman's bandwidth, PM and EM tests by means of real life data for the detection of unimodality, bimodality and multimodality. The paper is organized as follows: In Section 2, the methodology is given. In Sections 3-5, the applications with modality tests are presented and modality tests are compared.

2. MATERIALS AND METHODS

In this section, we introduce the steps of the methodology. After plotting the data, the shapes are analyzed by Kernel-Smoothing (K-S) density to determine whether they are unimodal, bimodal or multimodal. K-S density is used to estimate a probability density estimate of the sample in the vector X and f is the vector of density values evaluated at the points in X_i. This estimate is based on a normal Kernel function from a window parameter 'width' which is the function for the number of points in X. The density is evaluated at 100 equally spaced points that cover the range of the data in X. Then, the testing procedure for the four tests has been applied to detect the shape of the data. The critical value has been calculated from the Monte Carlo simulation technique by 10000 times. In the end, the shape of the data has been analyzed based on the comparison of the calculated value of each test statistic of four tests and estimates the simulated critical value of these tests. The inferences are presented to conclude whether the test follows the unimodal, bimodal or multimodal distribution. The procedure and conclusion of each test on different real-life data.

The critical value of Hartigan's dip, Silverman's bandwidth, PM and EM tests are calculated by the Monte Carlo simulation technique with Monte Carlo sample size 10,000. The test statistic value for each test is calculated. The graph of the real-life application reflects the range of values in X (data) along X-axis and Y-axis presents the number of elements within the groups.

2.1. Application of Modality Tests on Life Expectancy at Birth

The data of life expectancy at birth of Americans of both sexes between the years 1930 and 2010 collected from http://www.infoplease.com. The average life expectancy is 73.85 and the standard deviation is 4.147372. After plotting the data through K-S-density, the shape of the data looks like bimodal. In Figure 1, life expectancy at birth is presented.

The values of life expectancy at birth of both sexes in the USA from 1930 to 2010 are plotted along X-axis and Y-axis presents the number of elements within the groups. As seen in Figure 1, the graph looks bimodal. Then, we conduct a modality test and results are given in Table 1. Table 1 shows that Hartigan's dip test, Silverman's bandwidth

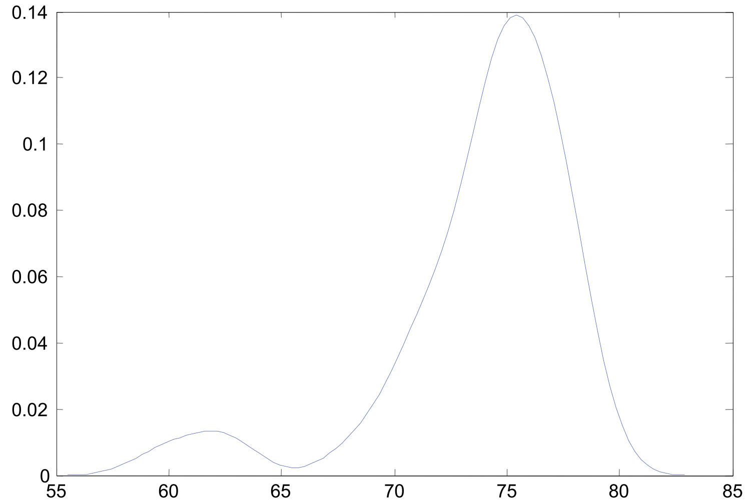


Fig. 1. Life expectancy at birth of both sexes at USA between years 1930 and 2010

Table 1. . Modality test results for the life expectancy at birth of both sexes at USA between years 1930 and 2010

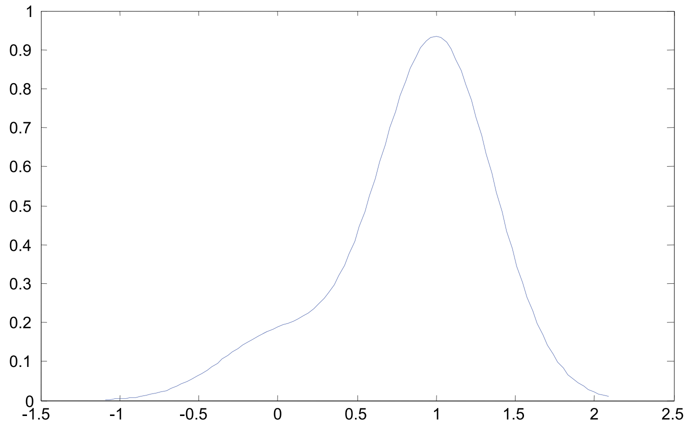
Life expectancy at birth of both sexes at USA (1930-2010)				
Tests	**Simulated Critical value**	**Calculated Value**	**Decision**	**Conclusion**
Hartigan's Dip Test	0.0128	0.0357	Calculated value is greater than Critical Value so reject Ho.	There is bimodality in the data.
Silverman's Bandwidth Test	0.0099	0.0181	Calculated value is greater than Critical Value so reject Ho.	There is bimodality in the data.
Proportional Mass test	0.5342	0.3133	Calculated value is less than Critical value so do not reject Ho.	There is unimodality in the data.
Excess Mass Test	0.0342	0.0714	Calculated value is greater than Critical value so reject Ho.	There is bimodality in the data.

test and EM test catch the bimodality, because of a relatively large sample size. However, the PM test shows that there is unimodality in the data, while the original data is bimodal. So, it is concluded that the PM test is appropriate only for a large sample, while Hartigan's dip and the EM tests work properly for relatively large sample size. However, the Silverman's bandwidth is the only test which is appropriate for detecting modality not only for the small and large sample but even for the small bump.

2.2. Application of Modality Tests on Poverty Status of Borrowers Before & after Availing Microfinance

Microfinance plays an important role in alleviating poverty and increasing economic empowerment among poor people. In this study, the data have been obtained from Aslam [1] to present the poverty status of borrowers before availing microfinance through the comparison of their income per household member with the poverty line given in the economic survey of Bahawalpur, Pakistan. Poverty status ranges from 0 to 1. The graph in Figure 2 shows unimodal shape by clustering on the value of 1 which means that the majority of borrowers are poor before availing microfinance.

The poverty status of borrowers before availing microfinance is plotted along X-axis (years) and Y-axis is the number of elements within the groups. Figure 2 shows that the graph presents a unimodal shape. Table 2 shows that all tests detect



(years)

Fig. 2. Poverty status of borrowers before availing microfinance in Bahawalpur

Table 2. Modality test results of poverty status of borrowers before availing microfinance

Poverty Status of Borrowers Before Availing Microfinance				
Tests	**Simulated Critical value**	**Calculated Value**	**Decision**	**Conclusion**
Hartigan's Dip Test	0.0834	0.0762	Calculated value is less than Critical value so do not reject Ho.	There is unimodality in the data.
Silverman's Bandwidth Test	0.0167	0.0044	Calculated value is less than Critical value so do not reject Ho.	There is unimodality in the data.
Proportional Mass test	1.4687	1.3730	Calculated value is less than Critical value so do not reject Ho.	There is unimodality in the data.
Excess Mass Test	0.1636	0.1524	Calculated value is less than Critical value so do not reject Ho.	There is unimodality in the data.

unimodality easily because the shape of the original data is unimodal. As the data is characterized by a comparatively large sample of more than 200, the entire four tests succeed in detecting unimodality which is originally present in the data.

As seen in Figure 3, the bimodal data of poverty status after availing microfinance shows the positive impact of microfinance on poverty alleviation. The major mode refers to non-poor and minor mode refers to the poor. The presence of the minor mode shows the partial effectiveness of microfinance on poverty alleviation. In Figure 3, the value of the poverty status of borrowers after availing microfinance along X-axis and Y-axis shows the number of elements within the groups. The graph in Figure 3 presents the bimodal shape.

The result in Table 3 shows that all the four tests detect bimodality easily because the shape of the original data is remarkably bimodal. As after availing microfinance, the many borrowers move from poor category to non-poor one so the presentation of data shows bimodality.

We conclude that before availing microfinance approximately all values tend to cluster around the value 1, representing poor, so all the four tests have detected unimodality easily because the shape of original data is unimodal. However, after availing the microfinance the values tend to cluster around two values 1 and 0 representing poor and non-poor respectively. As the data is characterized by clear bimodality and large sample size and graph has

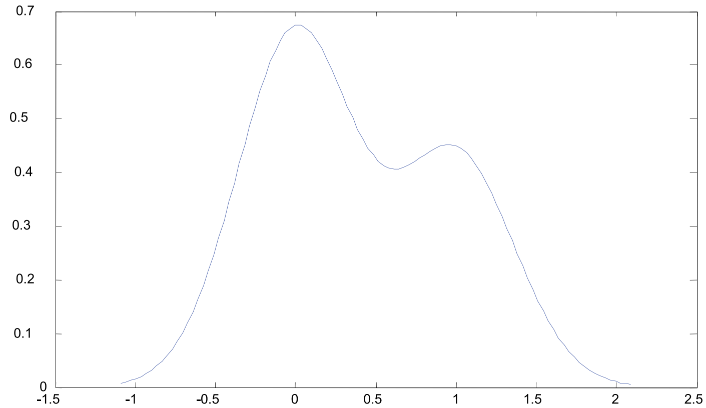


Fig. 3. The values of poverty status of borrowers after availing microfinance

Table 3. Modality test results of poverty Status of borrowers after availing microfinance

Poverty Status of Borrowers after Availing Microfinance				
Tests	**Simulated Critical value**	**Calculated Value**	**Decision**	**Conclusion**
Hartigan's Dip Test	0.2638	0.2976	Calculated value is greater than Critical value so reject Ho.	There is bimodality in the data.
Silverman's Bandwidth Test	0.0678	0.0942	Calculated value is greater than Critical value so reject Ho.	There is bimodality in the data.
Proportional Mass test	1.0457	1.0730	Calculated value is greater than Critical value so reject Ho.	There is bimodality in the data
Excess Mass Test	0.4061	0.4952	Calculated value is greater than Critical value so reject Ho.	There is bimodality in the data.

prominent bumps so all the four tests succeeded in detecting bimodality. Moreover, it is obtained that Silverman's bandwidth test is the most appropriate test as compared to other ones because it is applicable not only for both small and large sample size but also for both small and large bumps.

2.3. Application of Modality Tests on Point-Wise Ranking of Men's Hockey Teams of the World Data

The third data set is obtained from the internet, in which the positions-wise ranking points for the world men's field hockey have been assigned to 74 countries by the International Hockey Federation (IHF). The data shows that most of the countries cluster within 0 to 500 points, so the major mode lies within this range, some countries cluster between 1500 to 2000 points, this is minor mode and some countries cluster around points 2500 to 2800. The plot in Figure 4 shows that it is multimodal data of points-wise ranking of men's hockey teams of the world through K-S-density.

The points-wise ranking of men's hockey teams of the world data from 74 nations is plotted along X-axis and Y-axis shows the number of elements within the groups. The shape of the graph is multimodal. The graph in Figure 4 is also depicted through K-S-density. The data of points-wise ranking of men's hockey teams of the world is multimodal, but the three tests, Hartigan's dip test, PM test and EM test failed to detect multi-modality and concluded the presence of unimodality. The reason behind this result is a small bump. However, Silverman's test performs well. It is the most appropriate test as compare to other ones because it detects multimodality even with a small

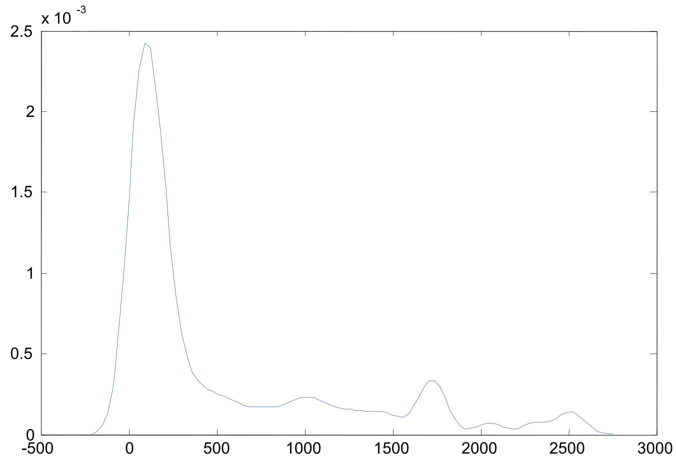


Fig. 4. Plot for positions-wise world data of men's field hockey teams

Table 4. Modality test results for the position-wise ranking of world data of men's field hockey teams.

Points wise ranking of men's hockey teams of the world data (74 nations)				
Tests	**Simulated Critical value**	**Calculated Value**	**Decision**	**Conclusion**
Hartigan's Dip Test	0.0234	0.0106	Calculated value is less than Critical value so do not reject Ho.	There is unimodality in the data
Silverman's Bandwidth Test	0.0019	0.0169	Calculated value is greater than Critical value so reject Ho.	There is multimodality in the data
Proportional Mass test	0.8962	0.8112	Calculated value is less than Critical value so do not reject Ho.	There is unimodality in the data
Excess Mass Test	0.0778	0.0612	Calculated value is less than Critical value so do not reject Ho.	There is unimodality in the data

sample or small bumps. It is overall concluded that Silverman's test is the more flexible test and it is the best of four tests for detecting modality whether it is unimodality, bimodality or multimodality.

3. CONCLUSIONS

From most of the real-life examples of modality tests, after the comparison of these tests, it has been concluded that the Hartigan's dip test, EM test work only for small samples and large bumps. However, the PM test only works for large sample and large bumps. So, Silverman's bandwidth test is only an appropriate test and flexible test which is appropriate for the small sample and small bumps and large samples, large bumps.

Overall comparison of modality tests on real-life situations, it is overall concluded that Silverman's bandwidth test is the more flexible test and it is the best of four tests for detecting modality whether it is unimodality, bimodality or multimodality.

4. REFERENCES

1. Aslam, S. Impact of Microfinance On Poverty Alliviation: A Case Study of Bahawalpur District, Allama Iqbal Open University Islamabad, PhD thesis (1999).
2. Bianchi, M. Testing for convergence: evidence from nonparametric multimodality tests, *Journal of Applied Econometrics,*12(4): 393-409 (1997).
3. Cheng, M, & P. Hall, Calibrating the excess mass and dip tests of modality, *Journal of the Royal*

Statistical Society, Series B, 60: 579-590(1998).

4. Cavallo, A, & R. Rigobon, The Distribution of the size of price changes, *NBER Working Paper No. 16760,* 1-40(2014).
5. Hall, P, & M. York, On the calibration of Silverman's test for multimodality. *Statistica Sinica*, 11: 515–536(2001).
6. Hartigan, P.M, Algorithm AS217: Computation of the Hartigan Dip Test Statistic to test for unimodality, *Appl. Statist,* 34: 320–325(1985).
7. Hartigan, P.M., & J.A Hartigan, the Hartigan Dip Test of unimodality, *Ann Statist*, 13(1): 70-84(1985).
8. Hall, P, & S. OOI, S, Attributing a Probability to the Shape of a Probability Density, *The Annals of Statistics, Institute of Mathematical Statistics,* 32(5): (2004).
9. Mueller, D.W. & G. Sawitzki, Excess mass estimates and tests for multimodality, *JASA*, 86: 738 -746(1991).
10. Silverman, B.W. *Some properties of a test for multimodality based on kernel density estimates, Probability, Statistics, and Analysis*, Cambridge University Press, Cambridge (1983).
11. Silverman, B.W, *Density Estimation for Statistics and Data Analysis, Monographs on Statistics and Applied Probability 26*, Chapman and Hall, London (1986).
12. Ling, X., E. Bedrick., T. Hanson., C. Restrepo, A Comparison of Statistical Tools for Identifying Modality in Body Mass Distributions, *Journal of Data Science*, 12:175-196(2014).