Pak. J. Engg. & Appl. Sci. Vol. 19 July, 2016 (*p.* 45–50)

Fourth Order Compact Method for One Dimensional Inhomogeneous Heat Equation

Zain Ul Abadin Zafar*[1], Anjum Pervaiz[2], M. Rafiq[1], M. Ozair Ahmad[2]

1. Faculty of Engineering, University of Central Punjab, Lahore, Pakistan.
2. Department of Mathematics, University of Engineering & Technology Lahore, Pakistan.

* **Corresponding Author:** E-mail: :*zainzafar2016@hotmail.com

Abstract

Many boundary value problems that come up in true life situations defy analytical solutions; so numerical techniques are the best source for finding the solution of such equations. In this paper, a compact method for inhomogeneous heat equation is developed. Comparison of the compact method with the second order scheme is also given. We obtain results both numerically and graphically. We used FORTRAN 90 for the calculations of the numerical results and MS office for the graphical comparison.

Key Words: Finite Difference Method, Fourth Order Compact Method, Linear Hyperbolic Equation.

1. Introduction

The compact method is a new difference approximation. It is the fourth order approximation using only three grid points; whereas standard fourth order centered difference approximation requires five points, so in this method we have a higher order approximation using fewer grid points. In (ORSZAG; 1974) a compact formula was mentioned. This method was used in that manner by Ciment and Leventhal (1978) for hyperbolic problems.

Let us take the second order one-dimensional linear equation

$$\frac{\partial u(x,t)}{\partial t} = c^2 \frac{\partial^2 u(x,t)}{\partial x^2} + p(x,t) \tag{1}$$

The initial condition is

$$u(x,0) = f(x) \tag{2}$$

and with the boundary conditions

$$u(0,t) = 0 \tag{3}$$

$$u(l,t) = 0 \tag{4}$$

for $0 \le x \le l \ , \ t > 0$

Eq. (1) is referred to as the second order inhomogeneous heat equation with constant coefficients. In eq. (1), x is distance and t is time, and $\beta, \ c^2$ are non negative integers.

2. Finite Difference Method

To set up the finite difference scheme for eq. (1), select an integer m and the values of t from 0 to ∞ then the mesh points (x_i, t_n) are

$$x_i = i\,\Delta\,x = ih \qquad \text{for } i = 0,1,2,3,\ldots m$$

$$t_n = n\,\Delta\,t = nk \qquad \text{for } n = 0,1,2,3,$$

At any interior mesh points (x_i, t_n), then the Inhomogeneous heat eq. (1) becomes

$$\beta \frac{\partial u(x_i,t_n)}{\partial t} = c^2 \frac{\partial^2 u(x_i,t_n)}{\partial x^2} + p(x_i,t_n) \tag{5}$$

The method is obtained using the central difference approximation for the 1st and 2nd order partial derivatives.

So that eq. (5) becomes

$$\frac{\beta}{2\,(\Delta t)}\left(u_i^{n+1} - u_i^{n-1}\right) - \frac{\beta(\Delta t)^2}{6}\frac{\partial^3 u(x_i,\mu_n)}{\partial t^3} = \frac{c^2}{(\Delta x)^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) - \frac{c^2(\Delta x)^2}{12}\frac{\partial^4 u(\xi_i,t_n)}{\partial x^4} + p_i^n$$

where $\xi_i = (x_i, x_{i+1})$

Neglecting the truncation error leads to the difference equation.

$$\frac{\beta}{2\,(\Delta t)}\left(u_i^{n+1}-u_i^{n-1}\right)=\frac{c^2}{(\Delta x)^2}\left(u_{i+1}^n-2u_i^n+u_{i-1}^n\right)+p_i^n$$

$$\frac{c^2}{(\Delta x)^2}\left(u_{i+1}^n+u_{i-1}^n\right)+p_i^n=\left(\frac{\beta}{2\,(\Delta t)}\right)u_i^{n+1}+\left(\frac{2c^2}{(\Delta x)^2}\right)u_i^n-\left(\frac{\beta}{2\,(\Delta t)}\right)u_i^{n-1}$$

Taking

$$\left(\frac{\beta}{2\,(\Delta t)}\right)=\lambda_1 \quad \text{and} \quad \left(\frac{2c^2}{(\Delta x)^2}\right)=\lambda_2.$$

So

$$\frac{c^2}{(\Delta x)^2}\left(u_{i+1}^n+u_{i-1}^n\right)+p_i^n=\lambda_1 u_i^{n+1}+\lambda_2 u_i^n-\lambda_1 u_i^{n-1}$$

$$\lambda_1 u_i^{n+1}=\frac{c^2}{(\Delta x)^2}\left(u_{i+1}^n+u_{i-1}^n\right)-\lambda_2 u_i^n-\lambda_1 u_i^{n-1}+p_i^n$$

$$u_i^{n+1}=\frac{c^2}{\lambda_1(\Delta x)^2}\left(u_{i+1}^n+u_{i-1}^n\right)-\frac{\lambda_2}{\lambda_1}u_i^n-u_i^{n-1}+\frac{1}{\lambda_1}p_i^n$$

By letting $\frac{c^2}{\lambda_1(\Delta x)^2}=\Lambda$, $\frac{-\lambda_2}{\lambda_1}=\Psi$, and $\frac{1}{\lambda_1}=\Omega$

So

$$u_i^{n+1}=\Lambda\left(u_{i+1}^n+u_{i-1}^n\right)+\Psi u_i^n-u_i^{n-1}+\Omega\, p_i^n$$

$$u_i^{n+1}=\Psi u_i^n+\Lambda\, u_{i+1}^n+\Lambda\, u_{i-1}^n-u_i^{n-1}+\Omega\, p_i^n \quad (6)$$

This equation holds for each $i=1,2,\ldots,(m-1)$. The boundary conditions give

$$u_0^n=u_m^n=0 \quad (7)$$

for each $n=1,2,\ldots$.

And the initial condition implies that

$$u_i^0=f(x_i) \quad (8)$$

for $i=1,2,\ldots,(m-1)$.

Writing in matrix form for $i=1,2,\ldots,(m-1)$, we have

$$\begin{bmatrix} u_1^{n+1}\\ u_2^{n+1}\\ \vdots\\ \vdots\\ u_{m-1}^{n+1}\end{bmatrix}=\begin{bmatrix}\Psi & \Lambda & 0 & & 0\\ \Lambda & \Psi & \Lambda & & \\ 0 & \Lambda & \ddots & \ddots & 0\\ & & \ddots & \Psi & \Lambda\\ 0 & & 0 & \Lambda & \Psi\end{bmatrix}\begin{bmatrix} u_1^{n}\\ u_2^{n}\\ \vdots\\ \vdots\\ u_{m-1}^{n}\end{bmatrix}-\begin{bmatrix} u_1^{n-1}\\ u_2^{n-1}\\ \vdots\\ \vdots\\ u_{m-1}^{n-1}\end{bmatrix}+\Omega\begin{bmatrix} p_1^{n}\\ p_2^{n}\\ \vdots\\ \vdots\\ p_{m-1}^{n}\end{bmatrix} \quad (9)$$

Equations (6) and (7) imply that the $(n+1)^{th}$ time steps requires values from the $(n)^{th}$ and $(n-1)^{th}$ time steps. This produces a minor starting problem since values of $n=1$ which is needed, in equation (6) to compute u_i^2 must be obtained from the initial value condition.

$$u_t|_i^0=g(x_i), \qquad 0\le x\le l$$

A better approximation $u_t|_i^0$ can be obtained rather easily, particularly when the second derivative of $'f'$ at $'x_i'$ can be determined.

Consider the Taylor Series

$$u_i^{n+1}=u_i^n+k\,u_t|_i^n+\frac{k^2}{2}\,u_{tt}|_i^n+\frac{k^3}{6}\,u_{ttt}|_i^n+\frac{k^4}{24}\,u_{tttt}|_i^n+\frac{k^5}{120}\,u_{ttttt}|_i^n+\cdots$$

$$\frac{u_i^{n+1}-u_i^n}{k}=u_t|_i^n+\frac{k}{2}\,u_{tt}|_i^n+\frac{k^2}{6}\,u_i^{(3)n}(x_i,\mu_n)$$

For $n=0$, we have

$$\frac{u_i^1-u_i^0}{k}=u_t|_i^0+\frac{k}{2}\,u_i^{(2)0}(x_i,\mu_n) \quad (10)$$

for some μ_n in $(0,t_1)$ and suppose the inhomogeneous heat equation also holds on the initial line. That is

$$u_t|_i^0=\frac{c^2}{\beta}\,f''(x_i)+\frac{1}{\beta}p_i^0$$

Substituting this value in eq.(10), we get

$$\frac{u_i^1-u_i^0}{k}=\left(\frac{c^2}{\beta}\,f''(x_i)+\frac{1}{\beta}p_i^0\right)+\frac{k}{2}\,u_i^{(2)0}(x_i,\mu_n)$$

So on simplifying we get

$$u_i^1=\frac{kc^2}{\beta}f''(x_i)+u_i^0+\frac{k}{\beta}p_i^0$$

This is an approximation with local truncation error $O(k^2)$ for each $i = 1,2,\dots,m-1$.

Now from the difference equation

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$$

$$u_i^1 = \frac{kc^2}{\beta h^2}(f(x_{i+1}) + f(x_{i-1})) + u_i^0 - \frac{2kc^2}{\beta h^2} f(x_i) + \frac{k}{\beta} p_i^0$$

But $u_i^0 = f(x_i)$ and letting $\lambda^2 = \frac{c^2}{h^2}$, we have

$$u_i^1 = \frac{k\,\lambda^2}{\beta}(f(x_{i+1}) + f(x_{i-1})) + \left(1 - \frac{2k\lambda^2}{\beta}\right) f(x_i) + \frac{k}{\beta} p_i^0 \tag{11}$$

for each $i = 1,2,\dots(m-1)$.

3. Compact Scheme for Inhomogeneous Heat Equation

To derive this method for the second order linear inhomogeneous heat eq. (1), with $f(x)$ and $g(x)$ are given functions. This Compact method approximates eq. (1) by two difference equations of fourth order using only three grid points say x_{i-1}, x_i and x_{i+1}. Let us denote 1st and 2nd derivatives of $u(x,t)$ w.r. to $'x'$ by F, S respectively.

$$u_x(x,t) = F \tag{12}$$

$$u_{xx}(x,t) = S$$

We shall initially derive a liaison between the values of F and u. Since $F = u_x$, it is clear that

$$u_{i+1}^n = u_{i-1}^n + \int_{i-1}^{i+1} F(\xi, l)\, d\xi$$

Approximating this integral by Simpson's Rule and after reorganize we get

$$u_{i+1}^n = u_{i-1}^n + \frac{h}{3}(F_{i-1}^n + 4F_i^n + F_{i+1}^n) + \frac{h^5}{90}\frac{\partial^4 F(\xi,l)}{\partial x^4}$$

Thus to fourth order, we have

$$(F_{i-1}^n + 4F_i^n + F_{i+1}^n) + \frac{3}{h}(u_{i-1}^n - u_{i+1}^n) = 0 \tag{13}$$

So we have a relationship between u and F. This is the first difference equation. In turn to get the second equation, we begin by evaluating (1) at the mid point $'i'$. Then eq. (1) becomes

$$\beta u_t|_i^n = c^2 S|_i^n + p_i^n \tag{14}$$

We at present want the expression for $S|_i^n$. If we articulate $u|_{i+1}^n$ and $u|_{i-1}^n$ in Taylor series about the point (i,n) and adding the result we get

$$u_{i+1}^n + u_{i-1}^n = 2u_i^n + h^2\, S|_i^n + \frac{h^4}{12} u_{xxxx}|_i^n + \frac{h^6}{360} u_{xxxxxx}(\xi, l)|_i^n \tag{15}$$

where we have replaced u_{xx} with $S|_i^n$. If we carry out the same procedure for F then we have

$$F_{i+1}^n - F_{i-1}^n = 2\,h\,S|_i^n + \frac{h^3}{3} u_{xxxx}|_i^n + \frac{h^5}{60} u_{xxxxx}(\xi, l)|_i^n \tag{16}$$

We now eliminate $u_{xxxx}|_i^n$ from these two equations and after rearranging, we get the following expression for $S|_i^n$, $S|_{i-1}^n$ and $S|_{i+1}^n$.

$$S|_i^n = \frac{2}{h^2}(u_{i+1}^n + u_{i-1}^n - 2u_i^n) - \frac{1}{2h}(F_{i+1}^n - F_{i-1}^n) + \frac{h^4}{360} u_{xxxxxx}(\xi, l)|_i^n$$

$$S|_{i-1}^n = \frac{1}{2h^2}(7u_{i+1}^n - 23u_{i-1}^n + 16u_i^n) - \frac{1}{h}(F_{i+1}^n + 6F_{i-1}^n + 8F_i^n) + \frac{h^4}{90} u_{xxxxxx}(\xi, l)|_i^n$$

And

$$S|_{i+1}^n = \frac{1}{2h^2}(7u_{i-1}^n - 23u_{i+1}^n + 16u_i^n) + \frac{1}{h}(F_{i-1}^n + 6F_{i+1}^n + 8F_i^n) + \frac{h^4}{90} u_{xxxxxx}(\xi, l)|_i^n$$

We now substitute the expression for $S|_i^n$ into (14) and rearrange to get the following second difference equation of fourth order.

$$\beta u_t|_i^n = \frac{2\,c^2}{h^2}(u_{i+1}^n + u_{i-1}^n) - \left(\frac{4\,c^2}{h^2}\right) u_i^n - \frac{c^2}{2h}(F_{i+1}^n - F_{i-1}^n) + p_i^n \tag{17}$$

We have now replaced (1) by two difference equations (13) and (17). Now we have to look at the boundaries. Let us first deem the left boundary condition i.e., at $x=0$ and denote the points $x=0,h,2h$ by 0,1,2. The first difference equation we obtain from the boundary condition is

$$u_0^n=0 \tag{18}$$

In order to obtain the second equation, we begin with the differential equation at the point 0 and 1.

$$c^2 S|_0^n+p_0^n=\beta u_t|_0^n \tag{19}$$

$$c^2 S|_1^n+p_1^n=\beta u_t|_1^n \tag{20}$$

So for $i=1$, we have the following expressions for $S|_0^n$ and $S|_1^n$.

$$S|_0^n=\frac{1}{2h^2}(-23u_0^n+16u_1^n+7u_2^n)-\frac{1}{h}(6F_0^n+8F_1^n+F_2^n) \tag{21}$$

$$S|_1^n=\frac{2}{h^2}(u_0^n-2u_1^n+u_2^n)-\frac{1}{2h}(F_2^n-F_0^n) \tag{22}$$

Finally we have from (13)

$$(F_0^n+4F_1^n+F_2^n)+\frac{3}{h}(u_0^n-u_2^n)=0 \tag{23}$$

So we have five equations (19) to (23). If we eliminate u_2^n, $S|_0^n$, $S|_1^n$ and F_2^n from these five equations, we get the second difference equation, valid at $x=0$.

$$\left(\frac{12c^2}{h^2}\right)u_0^n-\left(\frac{12c^2}{h^2}\right)u_1^n+\frac{6c^2}{h}F_0^n+\frac{6c^2}{h}F_1^n+(p_1^n-p_0^n)=\beta(u_t|_1^n-u_t|_0^n) \tag{24}$$

In a parallel way, we can develop the following difference equation for u and F at $x=m$, i.e. at the right boundary point.

$$u_m^n=0 \tag{25}$$

$$\left(\frac{12c^2}{h^2}\right)u_{m-1}^n-\left(\frac{12c^2}{h^2}\right)u_m^n+\frac{6c^2}{h}F_{m-1}^n+\frac{6c^2}{h}F_m^n+(p_m^n-p_{m-1}^n)=\beta(u_t|_m^n-u_t|_{m-1}^n) \tag{26}$$

Thus for each point, we have two difference equations. If we write them all together, we have the following Fourth Order Compact Scheme for u_{xx}.

$$u_0^n=0$$

$$\left(\frac{12c^2}{h^2}\right)u_0^n-\left(\frac{12c^2}{h^2}\right)u_1^n+\frac{6c^2}{h}F_0^n+\frac{6c^2}{h}F_1^n+(p_1^n-p_0^n)=\beta(u_t|_1^n-u_t|_0^n)$$

$$\frac{2c^2}{h^2}(u_{i+1}^n+u_{i-1}^n)-\left(\frac{4c^2}{h^2}\right)u_i^n-\frac{c^2}{2h}(F_{i+1}^n-F_{i-1}^n)+p_i^n=\beta u_t|_i^n$$

$$u_m^n=0$$

$$\left(\frac{12c^2}{h^2}\right)u_{m-1}^n-\left(\frac{12c^2}{h^2}\right)u_m^n+\frac{6c^2}{h}F_{m-1}^n+\frac{6c^2}{h}F_m^n+(p_m^n-p_{m-1}^n)=\beta(u_t|_m^n-u_t|_{m-1}^n)$$

The superscript n is used to denote the time grid lines.

4. Accuracy of the Scheme

Next, we compare the accuracy of the method with the standard five point centered difference scheme of the fourth order. The relation between F and u in this method is

$$\frac{1}{6}F_{i-1}+\frac{2}{3}F_i+\frac{1}{6}F_{i+1}=\frac{1}{2h}(u_{i+1}-u_{i-1})$$

and the relation between S and u in this method is

$$\frac{1}{12}S_{i-1}+\frac{5}{6}S_i+\frac{1}{12}S_{i-1}=\frac{1}{h^2}(u_{i+1}-2u_i+u_{i-1})$$

The correctness of this scheme is simply obtained by Taylor expansions of the above equations. The consequential truncation error is

$$F_i=u_i'-\left(\frac{1}{180}\right)h^4u^{(5)} \quad \text{and}$$

$$S_i=u_i''-\left(\frac{1}{240}\right)h^4u^{(6)}$$

The usual five-point fourth order approximations for u_x and u_{xx} are

$$F_i=\frac{1}{12h}(-u_{i+2}+8u_{i+1}-8u_{i-1}+u_{i-2}) \quad \text{and}$$

$$S_i=\frac{1}{12h^2}(-u_{i+2}+16u_{i+1}-30u_i+16u_{i-1}-u_{i-2})$$

The truncation error here is

$$F_i=u_i'-\left(\frac{1}{30}\right)h^4u^{(5)} \quad \text{and}$$

$$S_i=u_i''-\left(\frac{1}{90}\right)h^4u^{(6)}$$

Even though the new scheme and the standard representation both represent fourth order accuracy, the compact method should generate slightly more correct results due to smaller coefficients of the truncation error terms.

Difference scheme using compact scheme for u_{xx} and central difference scheme for u_t.

$$u_t|_i^n = \frac{u_i^{n+1} - u_i^{n-1}}{2k} + O(k^2)$$

Then

$$u_t|_i^n - u_t|_{i-1}^n = \frac{u_i^{n+1} - u_i^{n-1} - u_{i-1}^{n+1} + u_{i-1}^{n-1}}{2k}$$

Then by using the last results, we have from eqs. (18), (24), (23),(17), (25) and (26) as below:

$$u_0^n = 0$$

$$\left(\frac{\beta}{2k}\right)u_1^{n+1} - \frac{6c^2}{h}F_0^n - \frac{6c^2}{h}F_1^n = \left(-\frac{12c^2}{h^2}\right)u_1^n + \left(\frac{\beta}{2k}\right)u_1^{n-1} + (p_1^n - p_0^n)$$

$$F_{i-1}^n + 4F_i^n + F_{i+1}^n = \frac{3}{h}(u_{i+1}^n - u_{i-1}^n)$$

$$\left(\frac{\beta}{2k}\right)u_i^{n+1} + \frac{c^2}{2h}F_{i+1}^n - \frac{c^2}{2h}F_{i-1}^n = \frac{2c^2}{h^2}(u_{i+1}^n + u_{i-1}^n) + \left(-\frac{4c^2}{h^2}\right)u_i^n + \left(\frac{\beta}{2k}\right)u_i^{n-1} + p_i^n$$

$$u_m^n = 0$$

$$\left(\frac{\beta}{2k}\right)u_{m-1}^{n+1} + \frac{6c^2}{h}F_{m-1}^n + \frac{6c^2}{h}F_m^n = \left(-\frac{12c^2}{h^2}\right)u_{m-1}^n + \left(\frac{\beta}{2k}\right)u_{m-1}^{n-1} + (p_{m-1}^n - p_m^n)$$

Now for finding u_i^1 for the next time level. It can be approximated into the form by using Taylor's series and finite differences as given in eq.(11). The Fourth Order Scheme can be expressed in matrix form.

5. Test Problem

Consider the inhomogeneous heat equation $u_{xx} + (\pi^2 - 1)e^{-t}\sin\pi x = u_t$ in the interval $0 < x < 1$. The boundary conditions are

$$u(0,t) = u(1,t) = 0$$

and the initial conditions are

$$u(x,0) = \sin\pi x \quad , \quad 0 \le x \le 1, \quad t > 0.$$

The Exact Solution is $u(x,t) = e^{-t}\sin\pi x$.

Comparison of the Numerical Results of FDM and FOCM

Table 1 Finite Difference Method at $t = 0.02$

x_i	FDM	Exact	Error
0.000000000	0.000000000	0.000000000	0.000000000
0.100000000	0.303892718	0.302898048	0.000994670
0.200000000	0.578038301	0.576146324	0.001891977
0.300000000	0.795601468	0.792997385	0.002604083
0.400000000	0.935285617	0.932224336	0.003061281
0.500000000	0.983417496	0.980198673	0.003218823
0.600000000	0.935285617	0.932224336	0.003061281
0.700000000	0.795601468	0.792997385	0.002604083
0.800000000	0.578038301	0.576146324	0.001891977
0.900000000	0.303892718	0.302898048	0.000994670
1.000000000	0.000000000	0.000000000	0.000000000

Table 2 Fourth Order Compact Method at $t = 0.02$

x_i	FOCM	Exact	Error
0.000000000	0.000000000	0.000000000	0.000000000
0.100000000	0.302908939	0.302898048	0.000010891
0.200000000	0.576039594	0.576146324	0.000106730
0.300000000	0.792876389	0.792997385	0.000120996
0.400000000	0.932075901	0.932224336	0.000148435
0.500000000	0.980044480	0.980198673	0.000154193
0.600000000	0.932075901	0.932224336	0.000148435
0.700000000	0.792876389	0.792997385	0.000120996
0.800000000	0.576039594	0.576146324	0.000106730
0.900000000	0.302908939	0.302898048	0.000010891
1.000000000	0.000000000	0.000000000	0.000000000

For graph see Figure 1

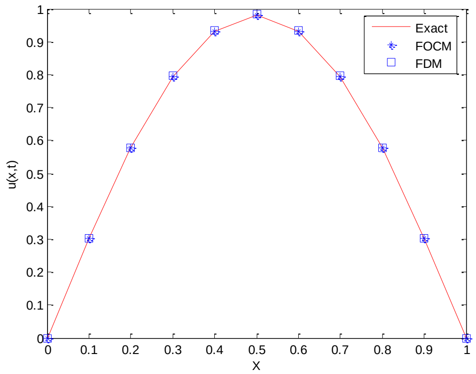

Figure 1

6. Conclusion

In this paper, numerical solutions of the one dimensional linear inhomogeneous heat equation are derived using Finite Difference Method (FDM) and Fourth Order Compact Method (FOCM). Fourth Order Compact Method is known to be a powerful device for solving functional equations. From the solutions of inhomogeneous heat equation, we note that the fourth order compact method, which also uses only three nodes, gives better results than the usual second order method.

7. References

[1] Orszag S. A. and M. ISRAELI, *Numerical Simulation of Viscous Incompressible Flows,* Annual Review of Fluid Mechanics by Milton Van Dyke, Vol. 6, Annual Reviews Inc., Palo Alto, California, 1974.

[2] Ciment M. Leventhal SH. *A note on the operator compact implicit method for the wave equation.* Math Comput. 1978; 32(1): 143-7.

[3] Abdul Majid Wazwaz, *Partial Differential Equations and Solitary Waves Theory,* Higher Education Press, Beijing 2009.

[4] Richard L. Burden, *Numerical Analysis,* 8th edition, Brooks Cole, 2004.

[5] Ozair, Muhammad Ahmad, *An Exploration of Compact Finite Difference Methods For the Numerical Solution of PDE*, Ph.D. Thesis, University of Western Ontario, Canada, 1997.