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Solution of a Vaccination Based SIR Epidemic Model by Homotopy Analysis Method

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Abstract

Modeling infectious diseases helped out to understand and overcome epidemics. This paper is based on epidemic model SIR, which fits well to many epidemiological diseases. Basic idea of Homotopy Analysis Method (HAM)is discussed and employed to compute an approximation to the solution of nonlinear system of differential equations. The effect of vaccination on the dynamics of childhood disease described by SIR model is monitored using HAM. The qualitative analysis reveals the vaccination reproduction number for disease control and eradication. MATLAB is used to carry out the computations. Graphical results are presented and discussed quantitatively.

Key Words: Epidemiology, Fixed Points**,** Homotopy, Reproduction Number**,** Stability

1. Introduction

When we talk about epidemic diseases then importance of vaccination could not be ruled out. When West recommended practicing vaccination (measles case) to all infants of 15 month age and repeat at the age of 6 approximately, it resulted in an increase in immunization. This recommendation is based on continuous-time constant vaccination strategy. Similarly period observations show that vaccination has increased the level of permanent immunity against epidemic disease [1]. Many infectious diseases caught children easier than others and named as childhood disease. Such diseases attack within age of 5 years. One fact is that in this period children are very much attached with their fellows, so disease spread rapidly. Some most common childhood diseases are measles, mumps, chicken pox, etc. Vaccination proved to be very effective against childhood disease [2]. To prevent the spread of childhood disease we need to germinate a structure that would foretell superfine vaccine coverage level.

2. SIR Model

SIR model is considered as a basic epidemic model. Most of the childhood diseases that propagate in population adjust quite simply into this model. Usually, diseases caused by a virus such as influenza, measles, and chickenpox, are of SIR type. This model was proposed by Kermack and Mckendrick in 1927. Many epidemiological diseases could be described by SIR model. Consider the flow of SIR model with constant vaccination strategy [3].

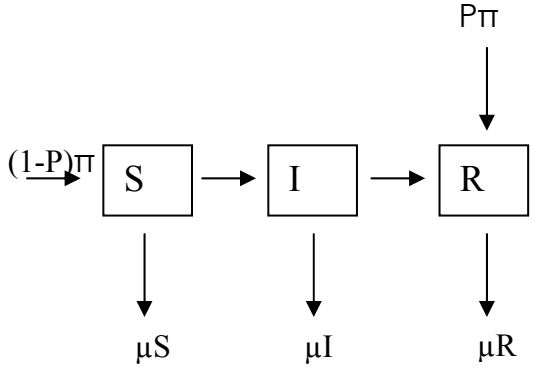

Fig.1 SIR Model with constant vaccination

where

S = Susceptible individuals
I = Infected
R = Recovered people with permanent immunity
μ = Natural death rate
β = average contact rate
R = Recover rate
π = Birth rate
P = new born vaccinated each year (0<P<1)

Where μ, β, γ and π are considered as positive parameters. Furthermore, we assumed that vaccination is 100% effective and the natural death rates μ and birth rate π are not same, this cause N to be not constant [4]. A susceptible will move to I-compartment when comes in contact with an infected individual, an infected individual move to R compartment after recovery. Vaccinated individuals are also coming into R-compartment. Now SIR model can be formulated as

$$\frac{dS}{dt} = (1-P)\pi N - \beta\frac{SI}{N} - \mu S \tag{1}$$

$$\frac{dI}{dt} = \beta\frac{SI}{N} - (\gamma+\mu)I \tag{2}$$

$$\frac{dR}{dt} = P\pi N + \gamma I - \mu R \tag{3}$$

We know $N = S + I + R$

Adding (1) to (3), we have

$$\frac{dN}{dt} = (\pi-\mu)N \tag{4}$$

We have a case of varying total population.

2.1 Dimensionless Transformation

We want to convert the varying total population into a constant total population, for this we have to choose new variables

$$s=\frac{S}{N}, \qquad i=\frac{I}{N}, \qquad r=\frac{R}{N}$$

$$s+i+r=1 \Rightarrow N=1$$

Now total population is constant i.e., $\frac{dN}{dt}=0$, from equation (4), we have birth rate equal to death rate

$$\pi = \mu$$

Putting respective values in (1), (2) & (3), new system is

$$\frac{ds}{dt} = (1-P)\pi - \beta si - \pi s \tag{5}$$

$$\frac{di}{dt} = \beta si - (\gamma+\pi)i \tag{6}$$

$$\frac{dr}{dt} = P\pi + \gamma i - \pi r \tag{7}$$

2.2 Qualitative Analysis

We will analyze system in two categories

1. Infection free equilibrium $(i=0)$
2. Endemic equilibrium $(i\neq 0)$

Subsystems in the closed set form are

$$\Gamma=\{(s,i)\in R+ \mid 0\le s+i\le 1\}$$

To find fixed points, from eq (5) & (6)

$$0=(1-p)\pi-\beta si-\pi s \tag{8}$$

$$0=\beta si-(\gamma+\pi)i \tag{9}$$

Case I. Infection free equilibrium

When disease dies out naturally then from eq (9)

$(\beta s-\gamma-\pi)\neq 0$ & $i=0$

From eq (8); $s=(1-P)$

The solution comes on an infection free equilibrium E_0 asymptotically

$$E_0=(1-P,0)$$

Reproduction number and basic reproduction numbers are:

$R_0=\frac{\beta}{\gamma+\pi}$ & $Rv=\frac{\beta(1-P)}{\gamma+\pi}$ respectively

This is a threshold which determines the stability of equilibrium.

Case II. Endemic Equilibrium

An unstable disease free equilibrium i.e., $Rv > 1$ give rise to endemic equilibrium E_u.

Again from eq (9);

$$0=(\beta s-\gamma-\pi)i$$

$$(\beta s-\gamma-\pi)=0 \qquad \& \qquad i\neq 0$$

$$\Rightarrow \qquad s=\frac{(1-P)}{Rv}$$

From of eq (8);

$$i=\frac{\pi}{\beta}\left(\frac{(1-P)}{s}-1\right) \quad or \quad i=\frac{\pi}{\beta}(Rv-1)$$

So, we have endemic equilibrium of the form

$$Eu = \left(\frac{(1-P)}{Rv}, \frac{\pi}{\beta}(Rv-1)\right)$$

2.3 Stability Analysis

The infection free equilibrium E_0 is locally stable if $Rv < 1$ and endemic equilibrium E_u is unstable [5]. Conversely for $Rv > 1$, endemic equilibrium E_u is stable and infection free equilibrium E_0 is unstable. In both cases local stability of equilibrium give rise to Global stability in the particular domain of *s* and *i* [6]. An examination of local stability of the model's equilibria reveals that there is a critical vaccination proportion

$$Pc = 1-\frac{1}{R0} \Rightarrow Pc - \frac{\beta-\gamma-\pi}{\beta}$$

P_c governs the system as follow

1. For relatively large vaccination level i.e., $Pc > P$, infection free equilibrium is locally stable with the coordinates

$$s=1-P \quad \& \quad i=0$$

While endemic equilibrium is unstable.

2. For relatively weak vaccination i.e., $Pc < P$, endemic equilibrium is locally stable with the coordinates

$$s=\frac{(1-P)}{Rv} \quad \& \quad \frac{\pi}{\beta}(Rv-1)$$

The Jacobian matrix at Endemic equilibrium Eu.

$$J=\begin{pmatrix} -\pi Rv & -(\gamma+\pi \\ \pi(Rv-1) & 0 \end{pmatrix}$$

$$trcJ = -\pi Rv$$

$$\det J = \pi Rv(\gamma+\pi)-\pi(\gamma+\pi)$$

As we know

$$trcJ \pm \frac{\sqrt{(trcJ)2-4(\det J)}}{2}$$

On putting values, we have

$$\lambda 1,2=\frac{-\pi Rv \pm \sqrt{(-\pi 2Rv)2-4[\pi Rv(\gamma+\pi)-\pi(\gamma+\pi)]}}{2}$$

For small values of π & γ, we neglect the last term under the square root sign

$$\lambda 1,2 \approx -\frac{\pi}{2}Rv \pm \frac{1}{2}\sqrt{\pi 2Rv_v^2 - 4Rv\pi(\gamma+\pi)}$$

For asymptotically stable, value under square root will be negative i.e.

$$Rv \le \frac{4(\gamma+\pi)}{\pi}$$

The endemic equilibrium E_u is locally asymptotically stable if

$$1 < Rv \le \frac{4(\gamma+\pi)}{\pi}$$

We have complex eigenvalues with negative real part. So *Eu* can be treated as a spiral sink. This can be explained as initially susceptible are increasing and we have few infected. Then infection starts spreading and susceptible start to decrease. Disease spread more rapidly than increment in susceptible. As a result we are left with too small number of individuals who are susceptible to disease, the outbreaks ends and susceptible begins to increase again.

3. Homotopy Analysis Method (HAM)

This new analytical technique was proposed by S.J. Liao in 1992. Homotopy Analysis Method is a general analytical approach use to solve nonlinear equations and solutions are obtained in the form of series [7]. HAM has a great potential to solve strongly nonlinear problems in science and engineering such as the viscous flows of non-Newtonian fluids, nonlinear heat transfer, finance problems, Riemann problems related to nonlinear shallow water equations, projectile motion, Glauert-jet flow, nonlinear water waves, groundwater flows and Laplace equations with certain boundary conditions. HAM is different than all perturbation and non-perturbation techniques because of the following facts.

1. Large or small parameters are of no significance in HAM

2. Convergence of solution can be ensured in a very simple way.
3. We are free to choose base function.

3.1 Underlying concept of HAM

We begin with a nonlinear algebraic equation

$$N[\,y(t)\,] = 0 \tag{10}$$

N is operator which is nonlinear and $y(t)$ exact solution is a function of the independent variable 't'. To construct homotopy we assume $sy_0(t)$ as the initial value of $y(t)$ and L is the auxiliary linear operator on the exact solution $y(t)$ such that $L[y(t)]=0$ when $y(t)=0$. We construct such a homotopy [8].

$$(1-q)\,L[\,y(t)-yo(t)]+qN[\,y(t)]=\hat{H}(t;q)$$

$q\in[0,1]$ is called the homotopy parameter. We are free to choose the initial value $yo(t)$ and operator L. Enforcing the homotopy to be zero i.e., $\hat{H}(t;q)=0$, we have

$$(1-q)\,L[\,y(t)-yo(t)]+qN[\,y(t)]=0$$

Above equation is not a single algebraic equation, it is a family. Homotopy parameter q plays an important role to solve it, now we can write above family as

$$(1-q)\,L[\varphi(t;q)-yo(t)]+qN[\varphi y(t;q)]=0 \tag{11}$$

From eq (11) we find that as q increases from 0 to 1, $\varphi(t;q)$ continuously changes from $yo(t)$ to the $y(t)$ of eq(10). It is continuous deformation, called Homotopy. Equation (11) is called deformation equation of order zero because of $\varphi(t;q)$, Now we use Maclaurin series to expand it

$$\varphi(t;q)=yo(t)+\sum_{m=1}^{\infty} ym(t)qm \tag{12}$$

where $\varphi(t;0)$ is employed and

$$y_m(t)=\frac{1}{m!}\,\frac{\partial m}{\partial qm}\varphi(t;q)_{q=0}=D_m(\varphi)$$

We select initial approximation $y_0(t)$ and the auxiliary linear parameter L in such a way that:

1. The solution $\varphi(t;q)$ of the zero order deformation equation exist for all $q\in$ [0, 1].
2. The deformation derivative $\dfrac{\partial m}{\partial q^m}\varphi(t;q)\Big|_{q=0}$ exist for m = 1, 2, …
3. The power series of $\varphi(t;q)$ converges at q = 1.

So the solution series

$$\varphi(t;q)=uo(t)+\sum_{m=1}^{\infty} ym(t)$$

or $$y(t)=\sum_{m=0}^{\infty} y_m(t)$$

for briefness , we write in form of vector a

$$\vec{y}(t)=\{y0(t),y1(t),\quad y2(t)\ldots\ ym(t)\}$$

This analytical approach is liberated from any physical parameters; HAM is strong enough that its efficiency will not affect whether a nonlinear equation contains small physical parameter or large. Equations like (11) are not always convergent at q=1, it might be divergent. To overcome this difficulty Liao introduced an auxiliary parameter $h\neq 0$ so zeroth order deformation equation becomes

$$(1-q)L[\varphi(t;q)-yo(t)]-qhH(t)N[\varphi(t;q)]=0 \tag{13}$$

where H(t) is the auxiliary function independent of q and we have the great freedom to choose it. Now the modified equation (13) is our zero order deformation equation. Now we start differentiating equation (13) with respect to q;

$$-L[\varphi(t;q)-yo(t)]+(1-q)L\left[\frac{\partial\varphi(t;q)}{\partial q}-0\right]$$
$$=\frac{hH(t)\,N[\varphi(t;q)]\,qhH(t)\,\partial N}{\partial q}[\varphi(t;q)]$$
$$-2L\frac{\partial\varphi(t;q)}{\partial q}+(1-q)L\frac{\partial 2\varphi(t;q)}{\partial q2}=$$
$$=2hH(t)\,\frac{\partial N[\varphi(t;q)]}{\partial q}\,\frac{qhH(t)\,(\partial 2N[\varphi(t;q)])}{\partial q2}$$
$$-3L\frac{\partial 2\varphi(t;q)}{\partial q2}+(1-q)L\frac{\partial 3\varphi(t;q)}{\partial q3}=$$

$$= 3hH(t)\ \frac{\partial 2N[\varphi(t;q)]}{\partial q2}\ \frac{qhH(t)(\partial 3N[\varphi(t;q)])}{\partial q3}$$

At mth derivative

$$-mXmL\frac{\partial m-1}{\partial qm-1}\varphi(t;q)+(1-q)L\frac{\partial m}{\partial qm}\varphi(t;q)=$$

$$=mhH(t)\ \frac{\partial m-1}{\partial qm-1}N[\varphi(t;q)]qhH(t)\frac{\partial m}{\partial qm}N[\varphi(t;q)]$$

Where

$$Xm=\begin{cases}0, & if\ \ m\le 1\\ 1, & if\ \ m>1\end{cases}$$

Or

$$L\left[(1-q)\frac{\partial m}{\partial qm}\varphi(t;q)-mXm\frac{\partial m-1}{\partial qm-1}\varphi(t;q)\right]$$

$$=\frac{mhH(t)\partial 1}{\partial qm-1}N[\varphi(t;q)]+\frac{qhH(t)\partial m}{\partial qm}N[\varphi(t;q)]$$

Substitute q=0 and divide by m!;

$$L\left[\frac{1}{m!}\frac{\partial m}{\partial qm}\varphi(t;q)-\frac{1}{(m-1)!}\frac{\partial m-1}{\partial q-1}\varphi(t;q)\right]=\frac{1}{(m-1)!}hH(t)\partial m-1\ \frac{}{\partial qm-1}N[\varphi(t;q)]$$

We know $ym(t)=\frac{1}{m!}\frac{\partial m}{\partial qm}\varphi(t;q)$ in above equation,

we have

$$L[ym(t)-Xmym-1(t)]=hH(t)\Re m(ym-1(t)) \qquad (14)$$

Where

$$\Re m(ym-1(t))=\frac{1}{(m-1)!}\frac{\partial M-1}{\partial QM-1}N[\varphi(t;q)] \qquad (15)$$

$$Xm=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases}$$

Obviously the higher order deformation equation (14) is governing by the linear operator L, and the term $\Re m(ym-1(t))$ which can be expressed for any nonlinear operator N. According to definition (15), the right hand side of eq (14) is only dependent upon $ym-1(t)$. Thus, we gain $y1(t),\ y2(t)$. By mean of solving the linear higher order deformation equation (14) one by one in order.

4. Solution of Model

We have to solve the equations 5 to 10.

To solve the system by Homotopy analysis method we assume the continuous mapping

$$s(t)\to\varphi 1(t;q),\ i(t)\to\varphi' 2(t;q), r(t)\to\varphi' 3(t;q)$$

Now we chose the auxiliary operator as

$$Li[\varphi' i(t;q)]=\frac{\partial}{\partial t}\varphi' i(t;q) \qquad i=1,2,3$$

With the property

$Li(Ci)=0$ where Ci is constant integral

Defining non-linear operators N_1, N_2, N_3 for equation (5), (6) & (7)

$$N1[\varphi' 1(t;q)]=\frac{\partial}{\partial t}\varphi' 1(t;q)-(1-P)\pi+\beta\varphi' 1(t;q)\varphi' 2(t;q)+\pi\varphi' 1(t;q)$$

$$N2[\varphi' 2(t;q)]=\frac{\partial}{\partial t}\varphi' 2(t;q)-\beta\varphi' 1(t;q)\varphi' 2(t;q)+(\gamma+\pi)\varphi' 2(t;q)$$

$$N3[\varphi' 3(t;q)]=\frac{\partial}{\partial t}\varphi' 3(t;q)-P\pi-\gamma\varphi' 2(t;q)+\pi\varphi' 3(t;q)$$

Now we can write family of zero order deformation equations as follow

$$(1-q)L[\varphi' 1(t;q)-s0(t)]=qh1H1N1[\varphi' 1(t;q)] \qquad (16)$$

$$(1-q)L[\varphi' 2(t;q)-i0(t)]=qh2H2N2[\varphi' 2(t;q)] \qquad (17)$$

$$(1-q)L[\varphi' 3(t;q)-r0(t)]=qh3H3N3[\varphi' 3(t;q)] \qquad (18)$$

Subject to initial conditions

$$\varphi' 1(0;q)=s0,\qquad \varphi' 2(0;q)=i0,\qquad \varphi' 3(0;q)=r0$$

By Taylor's theorem, we expand $\varphi_i'(t;q)$ by a power series of the embedding parameter q as follows:

$$\varphi' 1(t;q)=s0(t)+\sum_{m=1}^{\infty} sm(t)qm$$

$$\varphi' 2(t;q)=i0(t)+\sum_{m=1}^{\infty} im(t)qm$$

$$\varphi' 3(t;q)=r0(t)+\sum_{m=1}^{\infty} rm(t)qm$$

where

$$sm(t) = \frac{1}{m!}\frac{\partial m}{\partial qm}\varphi' 1(t;q)_{q=0}$$

$$im(t) = \frac{1}{m!}\frac{\partial m}{\partial qm}\varphi' 2(t;q)_{q=0}$$

$$rm(t) = \frac{1}{m!}\frac{\partial m}{\partial qm}\varphi' 3(t;q)_{q=0}$$

We know that zero order deformation equations (16), (17) and (18) can be written as

$$L[sm(t) - Xmsm-1(t)] = h1H1(t)\Re m(sm-1(t))$$

$$L[im(t) - Xmim-1(t)] = h2H2(t)\Re m(im-1(t))$$

$$L[rm(t) - Xmrm-1(t)] = h3H3(t)\Re m(rm-1(t))$$

$$sm(0) = 0,\ sm(0) = 0,\ rm(0)$$

where

$$\Re m(sm-1(t)) = \frac{1}{(m-1)!}\frac{\partial m-1}{\partial qm-1}N1[\varphi' 1(t;q)]$$

$$\Re m(im-1(t)) = \frac{1}{(m-1)!}\frac{\partial m-1}{\partial qm-1}N2[\varphi' 2(t;q)]$$

$$\Re m(rm-1(t)) = \frac{1}{(m-1)!}\frac{\partial m-1}{\partial qm-1}N3[\varphi' 3(t;q)]$$

$$Xm = \begin{cases} 0, & m \le 1 \\ 1, & m > 1 \end{cases}$$

As $Li = \frac{\partial}{\partial t}$ then taking integration on both sides and substituting $hi = -1, Hi = 1$ in above equations, we have

$$sm(t) = Xmsm-1(t) - \int_o^t \Re m(sm-1(t))dt \qquad (19)$$

$$im(t) = Xmim-1(t) - \int_o^t \Re m(im-1(t))dt \qquad (20)$$

$$rm(t) = Xmrm-1(t) - \int_o^t \Re m(rm-1(t))dt \qquad (21)$$

as

$$\Re m(sm-1(t)) = \frac{1}{(m-1)!}\frac{\partial m-1}{\partial qm-1}N1[\varphi' 1(t;q)]$$

Putting value of N_1

$$\Re m(sm-1(t)) = \frac{1}{(m-1)!}\frac{\partial m-1}{\partial qm-1}$$

$$\left[\frac{\partial}{\partial t}\varphi 1(t;q) - (1-P)\pi + \beta\varphi 1(t;q)\varphi 2(t;q) + \pi\varphi 1(t;q)\right]$$

$$= s'm-1(t) + \beta(\varphi 1(t;q)im-1(t;q) + \varphi 2(t;q)sm-1(t;q))$$

Using initial conditions

$$\varphi' 1(0;q) = s0, \qquad \varphi' 2(0;q) = i0$$

and $\varphi' 1(t;q) \to s0(t), \quad \varphi' 2(t;q) \to i0(t)$

$$\Re m(sm-1(t)) = s'm-1(t) + \beta(s0im-1 + i0sm-1) + \pi sm-1(t)$$

$$\Re m(sm-1(t)) = s'm-1(t) + \beta\left[\sum_{k=0}^{m-1} sk(t)im-1-k(t)\right] + \pi sm-1(t)$$

So, equation (19) becomes

$$sm(t) = Xmsm-1(t)$$

$$-\int_0^t \left[s'm-1(t) + \beta\sum_{k=0}^{m-1} sk(t)im-1-k(t)\pi sm-1(t)\right]dt \qquad (22)$$

Similarly equation (20) & (21) becomes

$$im(t) = Xmim-1(t)$$

$$-\int_0^t \left[ism-1(t) - \beta\sum_{k=0}^{m-1} sk(t)im-1-k(t) + (\gamma+\pi)im-1(t)\right]dt \qquad (23)$$

$$rm(t) = Xmrm-1(t)$$

$$-\int_0^t [\dot{r}m-1(t) - \gamma im-1(t) + \pi rm-1]dt \qquad (24)$$

Equation (22), (23) & (24) form system of mth-order deformation equations for $m > 1$.

Now we start iterating the above system of equations for different values of parameters and with the help of these parameters, we will observe the effect of high or low vaccination on childhood disease, hence stability will be discussed. We will discuss four cases [3];

Table. 1 Initial values and parameters

Case	s_0	i_0	r_0	B	γ	Π	P	P_c
1	1	0	0	0.8	0.03	0.04	0.9	0.4625
2	0.8	0.2	0	0.8	0.03	0.04	0.9	0.4625
3	0.8	0.2	0	0.8	0.03	0.04	0.3	0.4625
4	0.8	0.2	0	0.8	0.03	0.04	0.0	0.4624

Case 1

$$Rv = \frac{\beta(1-P)}{\gamma+\pi} = 0.18604$$

Since $Rv < 1$, E_0 is stable, E_u is unstable and have disease eradication.

For t = 0,

$$\frac{ds}{dt}_{t=0} = (1-P)\pi - \beta s0 i0 - \pi s0$$

$$= -0.36 \Rightarrow s = -0.36t = s1$$

$$s(t) = 1.0 - 0.36t + 0.72\times10^{-1}t^2 - 0.96\times10^{-2}t^3$$

$$+ 0.96\times10^{-3}t^4 - 0.786\times10^{-4}t^5$$

$$- 0.5688888892\times10^{-6}t^6$$

$$i(t) = 0$$

$$r(t) = 0.36t - 0.72\times10^{-1}t^2 + 0.96\times10^{-2}t^3 -$$

$$0.96\times10^{-3}t^4 + 0.768\times10^{-4}t^5 - 0.512\times10^{-5}t^6$$

$$\frac{ds}{dt}_{t=0} = \beta s0 i0 - (\gamma+\pi) i0$$

$$\Rightarrow i = 0 = i_1$$

$$\frac{dr}{dt}_{t=0} = P\pi + \gamma i0 - \pi r0$$

$$\Rightarrow r = 0.36t = r_1$$

For

$s_2, s_3, s_4 \ldots\ldots$, i_2, i_3, i_4 and $r_2, r_3, r_4 \ldots\ldots$,

We will use equation (22), (23) and (24) respectively.

Table.2 Approximations for case-1

Itr	s(t)	I(t)	r(t)
0	1.0	0	0
1	–0.36	0	0.36
2	0.72×10^{-1}	0	-0.72×10^{-1}
3	-0.96×10^{3}	0	0.96×10^{3}
4	0.96×10^{3}	0	-0.96×10^{3}
5	-0.768×10^{-4}	0	0.768×10^{4}
6	$-0.5688888892\times10^{-6}$	0	-0.515×10^{-5}

Case 2

Since $Rv < 1$, E_0 is stable, E_u is unstable and have disease eradication.

$$s(t) = 0.8 - 0.408t + 0.1008t^2 - 8.224\times10^{-3}t^3 -$$
$$0.1811776\times10^{-2}t^4 + 0.2838500158\times10^{-3}t^5 -$$
$$0.4866281149\times10^{-4}t^6 - 0.1973168518$$
$$\times10^{-5}t^7 + 0.1567280763\times10^{-7}t^9 +$$
$$0.4557699387\times 10^{-9}t^9 -$$
$$0.1747626667\times10^{-11}t^{10}$$

$$i(t) =$$
$$0.2 + 0.42\times10^{-1}t - 0.2823\times10^{-1}t^2 -$$
$$0.11687\times10^{-2}t^3 + 0.2759918751\times10^{-2}t^4 -$$
$$0.3762609484\times10^{-3}t^5 + 0.4741940899$$
$$\times10^{-4}t^6 + 0.199013997\times10^{-5}t^7 -$$
$$0.1540349563\times10^{-7}t^9 - 0.4575903832\times$$
$$10^{-9}t^9 + 0.1747626667\times10^{-11}t^{10}$$

$$r(t) = 0.366t - 0.7257\times10^{-1}t^2 + 0.93937\times10^{-2}t^3 -$$
$$0.94814275\times10^{-3}t^4 + 0.9241093251\times$$
$$10^{-43}t^5 - 0.4445486401\times10^{-5}t^6 -$$
$$0.1697145904\times10^{-7}t^7 - 0.269312\times10^{-9}t^9 +$$
$$0.1820444445\times10^{-11}t^9$$

Case 3.

Since $Rv > 1$, , E_0 is stable, E_u is stable and we don't have disease eradication

$$s(t) = 0.8 - 0.168t + 0.336\times10^{-1}t^2 - 0.2464\times10^{-2}t^3 -$$
$$0.12521\times10^{-3}t^4 + 0.22308195\times10^{-5}t^5 -$$
$$0.1932440964\times10^{-3}t^6 - 0.7803698863\times10^{-4}t^7$$
$$+ 0.4251830616\times10^{-5}t^9 + 0.1094303622\times$$
$$10^{-5}t^9 - 0.0293723614\times10^{-7}t^{10}$$

Table.3 Approximations for case-2

Itr	$s(t)$	$i(t)$	$r(t))$
0	0.8	0.2	0
1	–0.408	0.42×10^{-1}	0.366
2	0.1008	-0.2823×10^{-1}	-0.7257×10^{-1}
3	-8.224×10^{-3}	-0.11697×10^{-2}	0.93937×10^{-2}
4	-0.1811776×10^{-2}	$0.2759918751\times10^{-2}$	-0.94814275×10^{-3}
5	$0.2838500158\times10^{-3}$	$-0.3762609484\times10^{-3}$	$0.9241093251\times10^{-4}$
6	$-0.4866281149\times10^{-4}$	$0.4741940899\times10^{-4}$	$0.4445486401\times10^{-5}$
7	$-0.1973168518\times10^{-5}$	0.199013997×10^{-5}	$-0.1697145904\times10^{-7}$
8	$0.1567280763\times10^{-7}$	$-0.1540349563\times10^{-7}$	-0.269312×10^{-9}
9	$0.4557699387\times10^{-9}$	$-0.4575903832\times10^{-9}$	$0.182044444\times10^{-11}$
10	$-0.1747626667\times10^{-11}$	$0.1747626667\times10^{-11}$	0

Table.4 Approximations for case-3

Itr	$s(t)$	$i(t)$	$r(t))$
0	0.8	0.2	0
1	–0.168	0.42×10^{-1}	0.126
2	0.336×10^{-1}	-0.903×10^{-2}	-0.2457×10^{-1}
3	-0.2464×10^{-2}	-0.7217×10^{-3}	0.31857×10^{-2}
4	-0.125216×10^{-3}	0.44919875×10^{-3}	-0.32398275×10^{-3}
5	0.22308159×10^{-5}	$-0.308446284\times10^{-4}t^{5}$	0.286138125×10^{-4}
6	-1932440964×10^{-3}	$0.1860204709\times10^{-3}t^{6}$	$0.1534736535\times10^{-5}$
7	$-0.7803698868\times10^{-4}$	$0.7869655811\times10^{-4}t^{7}$	$-0.6595694941\times10^{-6}$
8	$0.4251830616\times10^{-5}$	$-0.41594566\times10^{-5}t^{9}$	$-0.9237401597\times10^{-7}$
9	$0.1094303622\times10^{-5}$	$-0.1098674509\times10^{-5}t^{9}$	$0.4370887113\times10^{-9}$
10	$-0.293723614\times10^{-7}$	$0.293723614\times10^{-7}t^{10}$	0

Table.5 Approximations for case-4

Itr	$s(t)$	$i(t)$	$r(t))$
0	0.8	0.2	0
1	-0.48×10^{-1}	0.42×10^{-1}	0.6×10^{-2}
2	0	0.57×10^{-3}	-0.57×10^{-3}
3	0.416×10^{-3}	-0.4977×10^{-3}	0.817×10^{-4}
4	0.26864×10^{-4}	-0.1496125×10^{-4}	-0.1190275×10^{-4}
5	-0.7711584×10^{-5}	0.68491314×10^{-5}	0.86245251×10^{-6}
6	-2076444349×10^{-3}	0.19838225×10^{-3}	$0.3573295998\times10^{-5}$
7	$-0.1402303147\times10^{-3}$	0.141400832×10^{-3}	$-0.1170517333\times10^{-5}$
8	$0.1075760762\times10^{-4}$	$-0.1048829562\times10^{-4}$	-0.269312×10^{-6}
9	$0.4557699387\times10^{-5}$	$-0.4575903832\times10^{-5}$	$0.1820444445\times10^{-7}$
10	$-0.1747626667\times10^{-6}$	0.174762666×10^{-6}	0

$$i(t) = 0.2 + 0.42\times10^{-1}t - 0.903\times10^{-2}t^2 - 0.7217\times10^{-3}t^3 + 0.44919875\times10^{-3}t^4 - 0.308446284\times10^{-4}t^5 + 0.1860204709\times10^{-3}t^6 + 0.7869655811\times10^{-4}t^7 - 0.41594566\times10^{-5}t^9 - 0.1098674509\times10^{-5}t^9 + 0.293723614\times10^{-7}t^{10}$$

$$r(t) = 0.126t - 0.2457\times10^{-1}t^2 + 0.31857\times10^{-2}t^3 - 0.3239827\times10^{-3}t^4 + 0.286138125\times10^{-4}t^5 + 0.1534736535\times10^{-5}t^6 - 0.6595694941\times10^{-6}t^7 - 0.9237401597\times10^{-7}t^9 + 0.4370887113\times10^{-9}t^9$$

Case 4

Since $Rv > 1$, E_0 is stable, E_u is stable and we don't have have disease eradication.

$$s(t) = 0.8 - 0.48\times10^{-1}t + 0.416\times10^{-3}t^3 + 0.26864\times10^{-4}t^4 - 0.7711584\times10^{-5}t^5 - 0.2076444349\times10^{-3}t^6 - 0.1402303147\times10^{-3}t^7 + 0.1075760762\times10^{-4}t^9 + 0.4557699387\times10^{-5}t^9 - 0.1747626667\times10^{-6}t^{10}$$

$$i(t) = 0.2 + 0.42\times10^{-1}t + 0.57\times10^{-3}t^3 - 0.4977\times10^{-3}t^3 - 0.1496125\times10^{-4}t^4 + 0.68491314\times10^{-5}t^5 + 0.19838225\times10^{-3}t^6 + 0.141400832\times10^{-3}t^7 - 0.1048829562\times10^{-4}t^9 - 0.4575903832\times10^{-5}t^9 + 0.1747626667\times10^{-6}t^{10}$$

$$r(t) = 0.6\times10^{-2}t - 0.57\times10^{-3}t^2 + 0.817\times10^{-4}t^3 - 0.1190275\times10^{-4}t^4 + 0.86245251\times10^{-6}t^5 + 0.3573295998\times10^{-5}t^6 - 0.117051333\times10^{-5}t^7 - 0.269312\times10^{-6}t^9 + 0.1820444445\times10^{-7}t^9$$

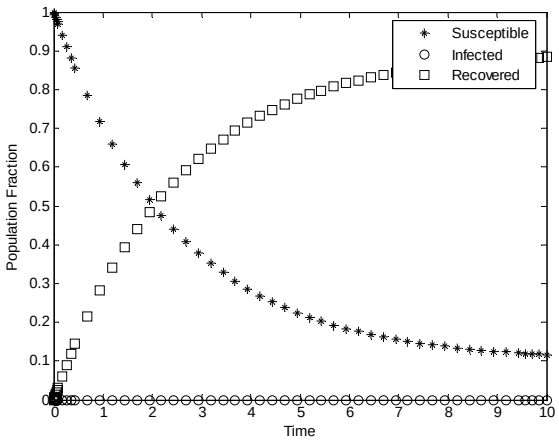

Fig.2 Plot for case 1

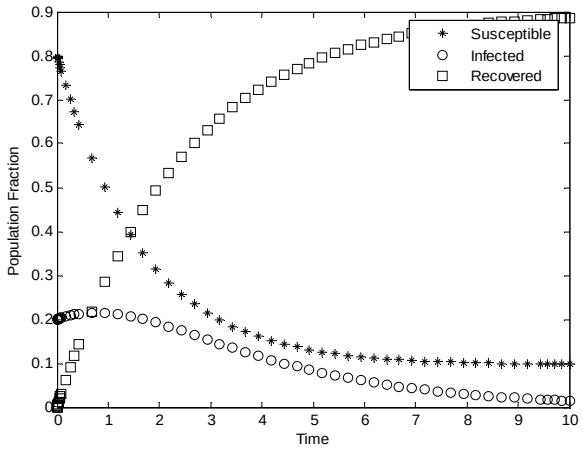

Fig.3 Plot for case 2

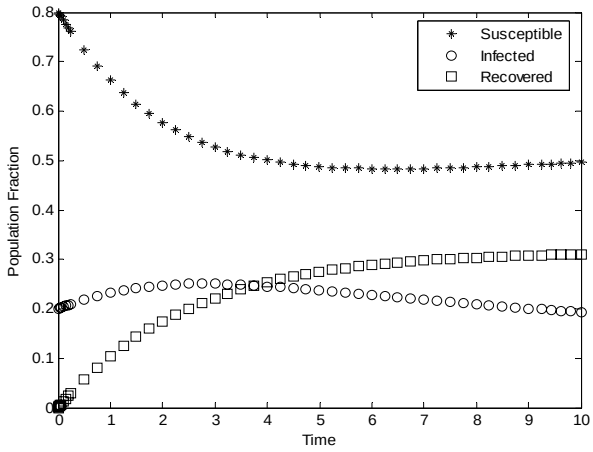

Fig.4 Plot for case 3

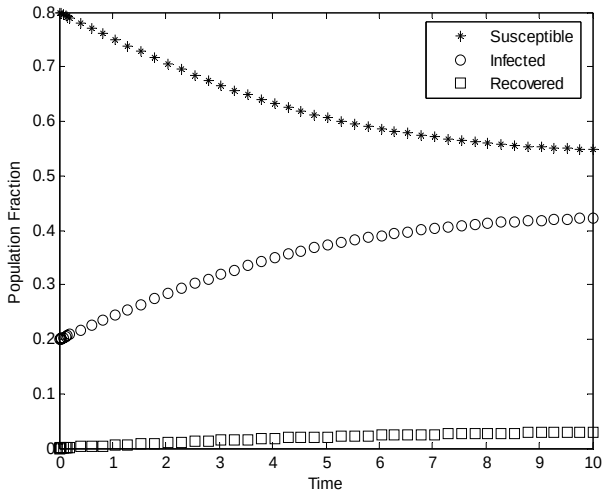

Fig.5 Plot for case 4

5. Results and Discussion

In first two cases high level of vaccination coverage is given and finally we reached infection free stage in both cases. In case 1, we did not introduce infection, all susceptible are given vaccination and all are recovered. In other words all susceptible moved to R-compartment without passing through I-compartment. Hence not a single infection is shown in Fig.2. While in case 2, we introduced small number of infection, two kinds of individuals are coming into R-compartment, one from S-compartment after being vaccinated and other from I-compartment with permanent immunity. In last two cases low level vaccination is implied as a result we have endemic situation at the end; disease permanently exists and infection free equilibrium cannot be attained.

It is clear that we can achieve infection free equilibrium if $(P > P_c)$ where P_c is threshold value, otherwise endemic situation occurs.

6. Conclusion

HAM generates series which converge speedily after some iteration. HAM nullifies much of computational work that arises in finite difference method and other parallel technique. Analytical approximations are time-tested and ensure the stability of HAM. Hence Homotopy Analysis Method (HAM) is reliable to solve non-linear system of differential equations.

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