Solution of Burger's Equation with the help of Laplace Decomposition Method

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Abstract

In this paper, the Laplace Decomposition Method (LDM) is employed to obtain approximate analytical solutions of the Burger Equation. The results show that the method converges rapidly and approximates the exact solution very accurately using only few iterates of the recursive scheme.

Key Words: Approximate solutions, Laplace Decomposition method, Adomian decomposition method, Burger's equation

1. Introduction

The Laplace Decomposition Method (LDM) is a numerical algorithm to solve nonlinear ordinary, partial differential equations. Khuri [1,2] used this method for the approximate solution of a class of nonlinear ordinary differential equations. So we are using same procedure on the partial differential equation. In this section we consider the non linear homogeneous Burger equation;

$$u_t = u_{xx} - uu_x \tag{1}$$

The initial condition is

$$u(x,0) = 1 - \frac{2}{x} \tag{2}$$

After applying the same procedure we have come to know that the results obtained are very much close to results as by using Adomian Decomposition Method.

2. Methodology

In this part we will use an Algorithm for Laplace Transform on the partial differential equations which are nonlinear. Here we will consider the general form of inhomogeneous nonlinear partial differential equations with initial conditions given below

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = h(x,t) \tag{3}$$

$$u(x,0) = f(x), u_t(x,0) = g(x) \tag{4}$$

where L is second order differential operator

$$L = \frac{\partial^2}{\partial t^2}$$

R is the remaining linear operator, $N\ u$ represents a general nonlinear differential operator and $h(x, t)$ is a source term. The first step we will take Laplace transform on equation (3).

$$\mathcal{L}\,[Lu(x,t)] + \mathcal{L}[R\,u(x,t)] + \mathcal{L}\,[N\,u(x,t)] = \mathcal{L}\,[h(x,t)] \tag{5}$$

By applying the Laplace transform differentiation property, we have

$$s^2\ \mathcal{L}\ [u(x,t)] - sf(x) - g(x) + [Ru(x,t)[+ \mathcal{L}[Nu(x,t)] = \mathcal{L}[h(x,t)] \tag{6}$$

$$\mathcal{L}[u(x,t)] = \frac{1}{S} f(x) + \frac{1}{S^2} g(x) + \frac{1}{S^2}\ \mathcal{L}[\mathrm{h(x,)}] - \frac{1}{s^2} \mathcal{L}[\mathrm{Ru(x,t)}] - \frac{1}{s^2}\ \mathcal{L}[\mathrm{Nu(x,t)}] \tag{7}$$

The second step is that we are going to represent the solution in an infinite series given below:

$$u = \sum_{n=0}^{\infty} u_n(x,t) \tag{8}$$

The nonlinear operator is written as

$$Nu(x,t) = \sum_{n=0}^{\infty} A_n(u) \tag{9}$$

where $A_n(u)$ are Adomian polynomials of $u_0, u_1, u_2, ..., u_n$ and it can be calculated by formula given below:

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} [N(\textstyle\sum_{i=0}^{\infty} \lambda^i u_i)]_{\lambda=0}, n = 0,1,2... \tag{10}$$

Substituting the equation (8), equation (9) and equation (10) in equation (7), which give us this result

$$\mathcal{L}\left[\sum_{n=0}^{\infty}u_n(x,t)\right]=\frac{1}{s}f(x)+\frac{1}{s^2}g(x)+\frac{1}{s^2}\mathcal{L}[h(x,t)]$$

$$-\frac{1}{s^2}\mathcal{L}[R\textstyle\sum_{n=0}^{\infty}u_n(x,t)]-\frac{1}{s^2}\mathcal{L}[\sum_{n=0}^{\infty}A_n(u)] \quad (11)$$

Or

$$\left[\sum_{n=0}^{\infty}\mathcal{L}\{u_n(x,t)\}\right]=\frac{1}{s}f(x)+\frac{1}{s^2}g(x)+\frac{1}{s^2}\mathcal{L}[h(x,t)]$$

$$-\frac{1}{s^2}\mathcal{L}[Ru(x,t)]-\frac{1}{s^2}\mathcal{L}[\textstyle\sum_{n=0}^{\infty}A_n(u)] \quad (12)$$

When we compare the left and right hand sides of equation (12) we obtain

$$\mathcal{L}[u_0(x,t)=\frac{1}{s}f(x)+\frac{1}{s^2}g(x)+\frac{1}{s^2}\mathcal{L}[h(x,t)] \quad (13)$$

$$\mathcal{L}[u_1(x,t)=-\frac{1}{s^2}\mathcal{L}[Ru_0(x,t)]-\frac{1}{s^2}\mathcal{L}[A_0(u)] \quad (14)$$

$$\mathcal{L}[u_2(x,t)=-\frac{1}{s^2}\mathcal{L}[Ru_1(x,t)]-\frac{1}{s^2}\mathcal{L}[A_0(u)] \quad (15)$$

The recursive relation, in general form is

$$\mathcal{L}[u_{n+1}(x,t)]=-\frac{1}{s^2}\mathcal{L}[Ru_n(x,t)]-\frac{1}{s^2}\mathcal{L}[A_n(u)] \quad (16)$$

Applying inverse Laplace transform to Eq. (13) to Eq. (16), so our recursive relation is as follows:

$$u_0(x,t)=K(x,t) \quad (17)$$

$$u_{n+1}(x,t)=-\mathcal{L}^{1}\left[\frac{1}{s^2}\mathcal{L}[Ru_n(x,t)]+\frac{1}{s^2}\mathcal{L}[A_n(u)]\right] \quad (18)$$

where $K(x,t)$ represents the term from source term and with the initial conditions. Now first of all, we are applying Laplace transform on the right hand side of Eq. (18) and then taking the inverse Laplace transform we get the values of $u_0, u_1, u_2, ..., u_n$, respectively.

3. Application 1

In this section we consider the homogeneous partial differential equation

$$u_t = u_{xx} - u u_x \quad (19)$$

The initial condition is

$$u(x,0)=1-\frac{2}{x} \quad (20)$$

Applying the algorithm of Laplace transform on equation (19), we have

$$s\mathcal{L}[u(x,t)]-u(x,0)=\mathcal{L}[u_{xx}(x,t)]-\mathcal{L}[u u_x] \quad (21)$$

Using given initial condition on Eq. (21) , we have

$$s\mathcal{L}[u(x,t)]=1-\frac{2}{x}+\mathcal{L}[u_{xx}(x,t)]-\mathcal{L}[u u_x] \quad (22)$$

Or

$$u(x,s)=\frac{1}{s}\left(1-\frac{2}{x}\right)+\frac{1}{s}\mathcal{L}[u_{xx}(x,t)]-\frac{1}{s}\mathcal{L}[u u_x] \quad (23)$$

Then applying the inverse Laplace transform to Eq. (23), we get

$$u(x,t)=\mathcal{L}^{1}\left[\frac{1}{s}\left(1-\frac{2}{x}\right)\right]+\mathcal{L}^{1}\left[\frac{1}{s}\mathcal{L}[u_{xx}(x,t)]\right]-$$

$$\mathcal{L}^{1}\left[\frac{1}{s}\mathcal{L}[u u_x]\right] \quad (24)$$

The Laplace Decomposition Method (LDM) assumes a series solution of the function $u(x,t)$ which is given by

$$u=\textstyle\sum_{n=0}^{\infty}u_n(x,t) \quad (25)$$

The non linear term is handled with the aid of Adomian Polynomials as below

$$u u_x=\textstyle\sum_{n=0}^{\infty}A_n(u) \quad (26)$$

Substituting equation (25) and equation (26) in equation (24), we have

$$u(x,t)=$$

$$\mathcal{L}^{1}\left[\frac{1}{s}\left(1-\frac{2}{x}\right)\right]+\mathcal{L}^{1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}\left(\textstyle\sum_{n=0}^{\infty}u_n(x,t)\right)\right]\right]-$$

$$\mathcal{L}^{1}\left[\frac{1}{s}\mathcal{L}[\textstyle\sum_{n=0}^{\infty}A_n(u)]\right] \quad (27)$$

The recursive relation is as follows:

$$u_0(x,t)=\left(1-\frac{2}{x}\right) \quad (28)$$

$$u_1(x,t)=-\mathcal{L}^{1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_0(x,t)\right]+\frac{1}{s}\mathcal{L}[A_0(u)]\right] \quad (29)$$

$$u_2(x,t)=-\mathcal{L}^{1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_1(x,t)\right]+\frac{1}{s}\mathcal{L}[A_1(u)]\right] \quad (30)$$

Similarly we have

$$u_{n+1}(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_n(x,t)\right] + \frac{1}{s}\mathcal{L}[A_n(u)]\right] \quad (31)$$

Also the non linear terms are of the form

$$A_0 = u_0 u_{0_x}$$

$$A_1 = u_{0_1} u_1 + u_0 u_{1_x}$$

$$A_2 = u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0$$

$$A_3 = u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{2_x} u_0$$

Therefore

$$A_o = u_0 u_{0_x} = \left(1 - \frac{2}{x}\right)\left(\frac{2}{x^2}\right) = \frac{2}{x^2} - \frac{4}{x^3}$$

Now

$$u_1(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}\left(-\frac{2}{x}\right)\right] + \frac{1}{s}\mathcal{L}\left[\frac{2}{x^2} - \frac{4}{x^3}\right]\right] \quad (31)$$

$$u_1(x,t) = -\frac{2}{x^2}t - \frac{4}{x^3}t + \frac{4}{x^3}t = -\frac{2}{x^2}t$$

$$A_1 = \left(\frac{2}{x^2}\right)\left(-\frac{2}{x^2}t\right) + \left(1 - \frac{2}{x}\right)\left(\frac{4}{x^3}t\right)$$

$$A_1 = \left(\frac{4}{x^3}t\right) - \left(\frac{12}{x^4}t\right)$$

$$u_2(x,t) = -\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}\left(-\frac{2}{x^2}t\right)\right] + \frac{1}{s}\mathcal{L}\left[\left(\frac{4}{x^3}t\right) - \left(\frac{12}{x^4}t\right)\right]\right]$$

$$u_2(x,t) = -\frac{6}{x^4}t^2 - \frac{2}{x^3}t^2 + \frac{6}{x^4}t^2 = -\frac{2}{x^3}t^2$$

The first few terms of $u_v(x,t)$ follows immediately

$$u(x,t) = 1 - \frac{2}{x} - \frac{2}{x^2}t - \frac{2}{x^3}t^2$$

Similarly, u_3, u_4, u_5, …, etc. can be obtained. Hence, all components of the decomposition (LDM) are identified. The complete solution is

$$u = \sum_{n=0}^{\infty} u_n$$

$$u(x,t) = 1 - \frac{2}{x}\left(1 + \frac{t}{x} + \frac{t^2}{x^2} + \ldots\right)$$

$$u(x,t) = 1 - \frac{2}{x}\left(1 - \frac{t}{x}\right)^{-1}$$

$$u(x,t) = 1 - \frac{2}{x\left(1 - \frac{t}{x}\right)}$$

$$u(x,t) = 1 - \frac{2}{(x-t)}$$

This is our required result in analytical form.

Application 2

In this section w consider the homogeneous partial differential equation

$$u_t = u_{xx} - uu_x \quad (32)$$

The initial condition is

$$u(x,0) = x \quad (33)$$

Applying the algorithm of Laplace transform on equation (32), we get

$$s\mathcal{L}[u(x,t)] - u(x,0) = \mathcal{L}[u_{xx}(x,t)] - \mathcal{L}[uu_x] \quad (34)$$

Using given initial condition on Eq. (21) becomes

$$s\mathcal{L}[u(x,t)] = x + \mathcal{L}[u_{xx}(x,t)] - \mathcal{L}[uu_x] \quad (35)$$

Or

$$u(x,s) = \frac{1}{s}x + \frac{1}{s}\mathcal{L}[u_{xx}(x,t)] - \frac{1}{s}\mathcal{L}[uu_x] \quad (36)$$

Then applying the inverse Laplace transform to Eq. (36), we get

$$u(x,t) = \mathcal{L}^{-1}[\tfrac{1}{s}x] + \mathcal{L}^{-1}\left[\tfrac{1}{s}\mathcal{L}[u_{xx}(x,t)]\right] - \mathcal{L}^{-1}\left[\tfrac{1}{s}\mathcal{L}[u\, u_x]\right] \quad (37)$$

The Laplace Decomposition Method (LDM) assumes a series solution of the function $u(x,t)$ is given by

$$u = \sum_{n=0}^{\infty} u_n(x,t) \quad (38)$$

The non linear term is handled with the aid of Adomian Polynomials as below

$$u u_x = \sum_{n=0}^{\infty} A_n(u) \tag{39}$$

Substituting equation (38) and equation (39) in equation (37), we have

$$u(x,t) = \mathcal{L}^{-1}[x] + \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} u_n(x,t)\right)\right]\right] -$$

$$\mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}[\sum_{n=0}^{\infty} A_n(u)]\right] \tag{40}$$

The recursive relation is as follows:

$$u_0(x,t) = x \tag{41}$$

$$u_1(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_0(x,t)\right] + \frac{1}{s}\mathcal{L}[A_0(u)]\right] \tag{42}$$

$$u_2(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_1(x,t)\right] + \frac{1}{s}\mathcal{L}[A_1(u)]\right] \tag{43}$$

Similarly we have

$$u_{n+1}(x,t) = \mathcal{L}^{-1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}u_n(x,t)\right] + \frac{1}{s}\mathcal{L}[A_n(u)]\right] \tag{44}$$

Also the non linear terms are of the form

$$A_0 = u_0 u_{0_x}$$
$$A_1 = u_{0_x} u_1 + u_0 u_{1_x}$$
$$A_2 = u_{0_x} u_2 + u_{1_x} u_1 + u_{2_x} u_0$$
$$A_3 = u_{0_x} u_3 + u_{1_x} u_2 + u_{2_x} u_1 + u_{2_x} u_0$$

Therefore

$$A_0 = u_0 u_{0_x} = x, 1 = x$$

Now

$$u_1(x,t) = -\mathcal{L}^{1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}x\right] + \frac{1}{s}\mathcal{L}[x]\right]$$

$$u_1(x,t) = -\mathcal{L}^{1}\left[\frac{1}{s}\mathcal{L}[0] + \frac{1}{s}\mathcal{L}[x]\right] = -xt$$

$$A_1 = (-xt)(1) + (x)(-t)$$

$$A_1 = -2xt$$

$$u_2(x,t) = -\mathcal{L}^{1}\left[\frac{1}{s}\mathcal{L}\left[\frac{\partial^2}{\partial x^2}(-2xt)\right] + \frac{1}{s}\mathcal{L}[-2xt]\right]$$

$$u_2(x,t) = -\mathcal{L}^{1}\left[\frac{1}{s}\mathcal{L}[0] + \frac{1}{s}\mathcal{L}[-2xt]\right] = 2xt^2$$

The first few terms of $u_n(x,t)$ follows immediately

$$u(x,t) = x - xt + 2xt^2$$

Similarly, $u_3, u_4, u_5, \ldots$ etc. can be obtained. Hence, all components of the decomposition (LDM) are identified. The complete solution is

$$u = \sum_{n=0}^{\infty} u_n$$
$$u(x,t) = x(1 - t + 2t^2 - \ldots)$$
$$u(x,t) = x(1+5)^{-1}$$
$$u(x,t) = \frac{x}{(1+t)}$$

This is our required result in analytical form.

4. Results and Discussion

In this paper, we have successfully developed LDM for solution of Burger's Equation. By keenly observing we have come up the result that LDM is an extremely powerful and efficient method in finding analytical solutions for a number of nonlinear problems.

6 References

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