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ZZ Fourth Order Compact BVM for the Equation of Lateral Heat Loss

Zain Ul Abadin Zafar[*,1], M.O. Ahmad, A. Pervaiz[1], Nazir Ahmad[1]

1. Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan.
* Corresponding Author: Zain_zafar1964@yahoo.com

Abstract

In this paper we combine the boundary value method (for discretizing the temporal variable) and finite difference scheme (for discretizing the spatial variables) to numerically solve the Equation of Lateral Heat Loss. This equation is also used in Probability, Stochastic processes and Brownian movements of gases. We first employ a fourth order compact scheme to discretize the spatial derivatives, and then a linear system of ordinary differential equations is obtained. Then we apply a fourth order scheme of boundary value method to approach this system. After this we use the central difference scheme for the temporal variables. Therefore, this scheme can achieve fourth order accuracy for spatial variables. For stability analysis we have used the Von Neumann Stability. Numerical results are presented to illustrate the accuracy and efficiency of this compact difference scheme, compared with finite difference scheme.

Key Words: Equation of lateral heat loss; Finite Difference Method; Fourth Order Compact Method

1. Introduction

Let us consider the equation with lateral heat loss in one dimensional as

$$\frac{\partial u(x,t)}{t} + c\,u\,)x,t) = a\frac{\partial^2 u(x,t)}{\partial x^2} \qquad (1)$$

with the following initial conditions

$$i(x,0) = f(x), x e \Omega \qquad (2)$$

and with the homogeneous Dirichlet boundary conditions

$$u(0,t) = 0 \qquad (3)$$

$$u(l,t) = 0 \quad for \;\; 0 \le x \le l, \quad t > 0$$

where $\Omega e[0.l]$, $'f'$, is sufficiently smooth and its required higher derivatives exist. a, c are non negative constants.

The term $c\ u$ represents heat flow across the lateral boundary. The temperature $u\ (x, t)$ at any point x_o is changing as a result of two phenomena.

(i) Diffusion of heat within the rod (due to $au_{x\,x}(x, t)$).

(ii) Heat flow across the lateral boundary (due to $l_c\ u\ (x, t)$).

The important point is that if there were no diffusion within the rod ($a = 0$), then the temperature at each point x_o would "damp" exponentially to zero.

2. Compact Method

Let us consider equation (1) with initial conditions given in equation (2) and homogeneous boundary conditions in (3) and (4).

Let $h = \frac{1}{N}$, be the uniform spatial mesh width. The spatial domain Ω can be subdivided by

$$x_i = i\Delta x = ih \quad \text{for } i = 0, 1, 2, 3, \ldots\ l.$$

We also discretize the temporal variable 't' by $t_n = nk$, $n = 0, 1, 2, \ldots, T$, where k is the step size in the temporal direction, T is the number of the time steps and the final time is $T_f = T\,k$.

We know by Taylor Series

$$u_i^{n+1} = u_i^n + k\,u_t\,\Big|_i^n + \frac{k^2}{2}u_{tt}\Big|_i^n + \frac{k^2}{6}u_{ttt}\Big|_i^n$$

$$+\frac{k^4}{24}u_{tttt}\Big|_i^n + \frac{k^5}{120}u_{ttttt}\Big|_i^n + \ldots. \qquad (5)$$

$$u_i^{n-1} = u_i^n - k\,u_t\ \Big|_i^n + \frac{k^2}{2}\,u_{tt}\Big|_i^n + \frac{k^2}{6}\,u_{ttt}\Big|_i^n$$

$$+\frac{k^4}{24}\,u_{tttt}\Big|_i^n - \frac{k^5}{120}\,u_{ttttt}\Big|_i^n + ... \qquad (6)$$

Subtracting equation (6) from (5) and dividing by 2 *k*.

$$\frac{u_i^{n+1} - u_i^{n-1}}{2k} = u_t\ \Big|_i^n + O(k^2)$$

Or $$\frac{1}{2k}\,\delta_t\,u_i^n = u_t\Big|_i^n + O(k^2) \qquad (7)$$

where $\delta_t\,u_i^n = u_i^{n+1} - u_i^{n-1}$

Similarly

$$\frac{1}{2h}\,\delta_x\,u_i^n = u_x\Big|_i^n + O(h^2) \qquad (8)$$

Where $\delta_x\,u_i^n = u_{i+1}^n - u_{i-1}^n$

Also we have

$$\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} = u_{xx}\Big|_i^n + \frac{h^2}{12}\,u_{xxxx}\ \Big|_i^n + O(h^4)$$

$$\frac{1}{h^2}\delta_x^2 u_i^n = u_{xx}\ \Big|_i^n + \frac{h^2}{12}\,u_{xxxx}\Big|_i^n + O(h^4) \qquad (9)$$

Where $\delta_x^2\,u_i^n = u_{i+1}^n - 2u_i^n + u_{i-1}^n$

Replace u_i^n by $u_t\Big|_i^n$ in equation (8) and (9).

$$\frac{1}{2h}\delta_x u_t\ \Big|_i^n = u_{xt}\ \Big|_i^n + O(h^2) \qquad (10)$$

$$\frac{1}{h^2}\delta_x^2 u_i = u_{xxt}\ \Big|_i^n + \frac{h^2}{12}\,u_{xxxxt}\Big|_i^n + O(h^4) \qquad (11)$$

By equation (1)

$$u_t + cu = au_{xx}$$

Operate by $\frac{\partial^2}{\partial x^2}$

$$\frac{\partial^2 u}{\partial x^2 \partial t} = a\frac{\partial^4 u}{\partial x^4} - c\frac{\partial^2 u}{\partial x^2}$$

Or

$$\frac{\partial^4 u}{\partial x^4} = \frac{1}{a}\,\frac{\partial^2 u}{\partial x^2 \partial t} + \frac{c}{a}\,\frac{\partial^2 u}{\partial x^2} \qquad (12)$$

But by equation (11)

$$\frac{1}{h^2}\delta_x^2\,u_t\ \Big|_i^n = u_{xxt}\ \Big|_i^n + O(h^2) \qquad (13)$$

Therefore equation (12) becomes

$$\frac{\partial^4 u}{\partial x^4}\ \Big|_i^n = \frac{1}{a}\left[\frac{1}{h^2}\delta_x^2\,u_t\ \Big|_i^n - O(h^2)\right] + \frac{c}{a}\,\frac{\partial^2 u}{\partial x^2}\Big|_i^n$$

$$\frac{\partial^4 u}{\partial x^4}\Big|_i^n = \frac{1}{a\,h^2}\delta_x^2\,u_t\Big|_i^n + \frac{c}{a}\,\frac{\partial^2 u}{\partial x^2}\Big|_i^n - O(h^2) \qquad (14)$$

Substitute this equation in equation (9), we have

$$\frac{1}{h^2}\delta_x^2 u_i^n = u_{xx}\Big|_i^n + \frac{h^2}{12}\left[\frac{1}{ah^2}\delta_x^2 u_t\ \Big|_i^n + \right.$$

$$\left.\frac{c}{a}\,\frac{\partial^2 u}{\partial x^2} - O(h_2)\ \right] + O(h^4)$$

$$\frac{1}{h^2}\delta_x^2 u_i^n = u_{xx}\Big|_i^n + \frac{1}{12a}\ \delta_x^2 u_i\ \Big|_i^n$$

$$+\frac{ch^2}{12\,a}\,\frac{\partial^2 u}{\partial x^2}\ \Big|_i^n + O(h^4)$$

$$\frac{1}{h^2}\delta_x^2 u_i^n = \left(1 + \frac{ch^2}{12\,a}\right)u_{xx}\Big|_i^n$$

$$+\frac{1}{12a}\ \delta_x^2 u_i\ \Big|_i^n + O(h^4) \qquad (15)$$

From equation (1) we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a}\,\frac{\partial u}{\partial t} + \frac{c}{a}\,u$$

Therefore equation (15) implies

$$\frac{1}{h^2}\delta_x^2 u_i^n = \left(1+\frac{ch^2}{12a}\right)\left(\frac{1}{a}\frac{\delta u}{\partial t}+\frac{c}{a}u\right)\Bigg|_i^n +$$

$$\frac{1}{12a}\delta_x^2 u_t\Big|_i^n + O(h^4) \qquad (15)$$

$$\frac{1}{h^2}\delta_x^2 u_i^n = \frac{1}{a}\left(1+\frac{ch^2}{12a}\right)\frac{\delta u}{\partial t}\Bigg|_i^n + \frac{c}{a}$$

$$\left(1+\frac{ch^2}{12a}\right) u\Big|_i^n + \frac{1}{12a}\delta_x^2 u_t \Big|_i^n + O(h^4)$$

$$\frac{1}{h^2}\delta_x^2 u_i^n = \frac{1}{a}\left(1+\frac{ch^2}{12a}-\frac{1}{6}\right)\frac{\delta u}{\partial t}\Bigg|_i^n + \frac{c}{a}$$

$$\left(1+\frac{ch^2}{12a}\right) u\Big|_i^n +$$

$$\frac{1}{12a}\left(u_t\Big|_{i+1}^n + u_t\Big|_{i-1}^n\right) + O(h^4) \qquad (16)$$

Or

$$\frac{1}{h^2}\left(u_{i+1}^n - 2u_i^n + u_{i-1}^n\right) = \frac{1}{a}\left(\frac{5}{6}-\frac{ch^2}{12a}\right)\frac{\delta u}{\partial t}\Bigg|_i^n$$

$$+\frac{c}{a}\left(1+\frac{ch^2}{12a}\right) u\Big|_i^n +$$

$$\frac{1}{12a}\left(u_t\Big|_{i+1}^n + u_t\Big|_{i-1}^n\right) + O(h^4)$$

$$\frac{1}{h^2}\left(u_{i+1}^n + 2u_{i-1}^n\right) - \frac{2}{h^2}u_i^n - \frac{c}{a}\left(1+\frac{ch^2}{12a}\right)$$

$$u_i^n = \frac{1}{a}\left(\frac{5}{6}+\frac{ch^2}{12a}\right) u_t\Big|_i^n +$$

$$\frac{1}{12a}\left(u_t\Big|_{i+1}^n + u_t\Big|_{i-1}^n\right) + O(h^4)$$

$$\frac{1}{h^2}\left(u_{i+1}^n + u_{i-1}^n\right) - \frac{2}{h^2}u_i^n - \frac{c}{a}\left(1+\frac{ch^2}{12a}\right)$$

$$u_i^n = \frac{1}{a}\left(\frac{5}{6}+\frac{ch^2}{12a}\right) u_t\Big|_i^n +$$

$$\frac{1}{12a}\left(u_t\Big|_{i+1}^n + u_t\Big|_{i-1}^n\right) + O(h^4)$$

After ignoring the truncation error

$$u_t\Big|_{i-1}^n + \frac{1}{a}\left(\frac{5}{6}+\frac{ch^2}{12a}\right)u_t\Big|_i^n + \frac{1}{12a}\ u_t\Big|_{i-1}^n$$

$$\frac{1}{h^2}u_{i-1}^n - \left(\frac{c}{a}+\frac{2}{h^2}+\frac{c^2h^2}{12a^2}\right)u_i^n + \frac{1}{h^2}u_{i+1}^n$$

$$= \frac{1}{12a}u_t\Big|_i^n + \frac{1}{a}\left(\frac{5}{6}+\frac{ch^2}{12a}\right) u_t\Big|_i^n +$$

$$\frac{1}{12a}\ u_t\Big|_{i+1}^n \qquad (17)$$

Let us take

$$u_t\Big|_i^n = \frac{u_i^{n+1} - u_i^{n-1}}{2k} - O(k^2)$$

Therefore equation (17) becomes

$$\frac{1}{h^2}u_{i-1}^n - \left(\frac{c}{a}+\frac{2}{h^2}+\frac{c^2h^2}{12a^2}\right)u_i^n + \frac{1}{h^2}u_{i+1}^n$$

$$= \frac{1}{12a}\left[\frac{u_{i-1}^{n+1} - u_{i-1}^{n-1}}{2k} - O(k^2)\right] +$$

$$\frac{1}{a}\left(\frac{5}{6}+\frac{ch^2}{12a}\right)\left[\frac{u_i^{n+1} - u_i^{n-1}}{2k} - O(k^2)\right] +$$

$$= \frac{1}{12a}\left[\frac{u_{i+1}^{n+1}-u_{i+1}^{n-1}}{2k} - O(k^2)\right]$$

After ignoring the truncation error

$$\frac{1}{h^2}u_{i-1}^n - \left(\frac{c}{a}+\frac{2}{h^2}+\frac{c^2h^2}{12a^2}\right)u_i^n + \frac{1}{h^2}u_{i+1}^n$$

$$= \frac{1}{12a}\left[\frac{u_{i-1}^{n+1}-u_{i-1}^{n-1}}{2k}\right]+$$

$$\frac{1}{a}\left(\frac{5}{6}+\frac{ch^2}{12a}\right)\left[\frac{u_i^{n+1}-u_i^{n-1}}{2k}\right]+$$

$$\frac{1}{12a}\left[\frac{u_{i+1}^{n+1}-u_{i+1}^{n-1}}{2k}\right]$$

$$\frac{1}{h^2}u_{i-1}^n - \left(\frac{c}{a}+\frac{2}{h^2}+\frac{c^2h^2}{12a^2}\right)u_i^n + \frac{1}{h^2}u_{i+1}^n$$

$$= \frac{1}{24ak}u_{i-1}^{n+1} - \frac{1}{24ak}u_{i-1}^{n-1} + \frac{1}{2ak}\left(\frac{5}{6}+\frac{ch^2}{12a}\right)$$

$$u_i^{n+1} - \frac{1}{2ak}\left(\frac{5}{6}+\frac{ch^2}{12a}\right)u_i^{n-1} + \frac{1}{24ak}1 \quad (18)$$

Let us take

$$\frac{1}{h^2} = A \quad -\left(\frac{c}{a}+\frac{2}{h^2}+\frac{c^2h^2}{12a^2}\right) = \psi$$

$$\frac{1}{24ak} = \phi \text{ and } \frac{1}{2ak}\left(\frac{5}{6}+\frac{ch^2}{12a}\right) = Y$$

Therefore equation (18) becomes

$$\wedge u_{i-1}^n + \psi u_i^n + \wedge u_{i+1}^n = \phi u_{i+1}^{n+1} + Yu_i^{n+1} +$$

$$+\phi u_{i+1}^{n+1} - \phi u_{i-1}^{n-1} - Yu_i^{n-1} - \phi u_{i+1}^{n-1} \quad (19)$$

Equation (19) can be written in matrix form for $i = 1,2,\dots,l-1$.

$$\begin{bmatrix} Y & \Phi & 0 & & 0 \\ \Phi & Y & \Phi & & 0 \\ 0 & \Phi & \ddots & \ddots & \\ & & \ddots & Y & \Phi \\ 0 & & 0 & \Phi & Y \end{bmatrix}\begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ \vdots \\ u_{i-1}^{n+1}\end{bmatrix} =$$

$$\begin{bmatrix} \Psi & A & 0 & & 0 \\ A & \Psi & A & & \\ 0 & A & \ddots & \ddots & 0 \\ & & \ddots & \Psi & A \\ 0 & & 0 & A & \Psi \end{bmatrix}\begin{bmatrix} u_1^{n} \\ u_2^{n} \\ \vdots \\ \vdots \\ u_{i-1}^{n}\end{bmatrix} +$$

$$\begin{bmatrix} Y & \Phi & 0 & & 0 \\ \Phi & Y & \Phi & & \\ 0 & \Phi & \ddots & \ddots & 0 \\ & & \ddots & Y & \Phi \\ 0 & & 0 & \Phi & Y \end{bmatrix}\begin{bmatrix} u_1^{n-1} \\ u_2^{n-1} \\ \vdots \\ \vdots \\ u_{i-1}^{n-1}\end{bmatrix} \quad (20)$$

Observing that the implicit scheme is three level in time. Since u_i^o is given, we need to evaluate u_i^1 for the next time level which can be evaluated as:

$$\frac{u_i^{n+1}-u_i^n}{k} = u_t\big|_i^n + \frac{k}{2}u_{tt}\big|_i^n + O(k^2)$$

If we replace n=0, then the above equation gives

$$\frac{u_i^1-u_i^o}{k} = u_t\big|_i^o + \frac{k}{2}u_{tt}\big|_i^o + O(k^2)$$

Or

$$\frac{u_i^1-u_i^o}{k} = u_t\big|_i^o + O(k^2)$$

$$u_i^1 - u_i^o = k\,u_t\big|_i^o + O(k^2)$$

$$u_i^1 - u_i^o + k\,u_t\big|_i^o + O(k^2)$$

But by equation (1) we have,

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} - cu$$

$$u_i^1 = u_i^o + k\left[a\frac{\partial^2 u}{\partial x^2} - cu\right]\Big|_i^o + O(k^2)$$

$$u_i^1 = u_i^o + ka\,u_{xx}\big|_i^o - ck\,u_i^o + O(k^2)$$

$$u_i^1 = (1-ck)u_i^o + ka\,u_{xx}\big|_i^o + O(k^2)$$

But

$$f''(x_i) = u_{xx}\big|_i^o =$$

$$\frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$$

And

$$u_i^o = f(x_i)$$

$$u_i^1 = (1-ch)\,f(x_i) + ka$$

$$\left[\frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}\right] + O(k^2)$$

$$u_i^1 = (1-ch)\,f(x_i) + \frac{ka}{h^2}(f(x_{i+1}) +$$

$$f(x_{i-1})) - \frac{2ka}{h^2}f(x_i) + O(k^2)$$

$$u_i^1 = \left(1 - ck - \frac{2ka}{h^2}\right)f(x_i) + \frac{ka}{h^2}(f(x_{i+1}) +$$

$$f(x_{i-1})) + O(k^2)$$

So after neglecting the truncation error

$$u_i^1 = \left(1 - ck - \frac{2ka}{h^2}\right)f(x_i) + \frac{ka}{h^2}$$

$$(f(x_{i+1}) + f(x_{i-1})) \qquad (21)$$

2. Finite Difference Scheme

To set up the finite difference scheme for eq. (1), select an integer l and the values of t from 0 to ∞ then the mesh points (x_i, t_n) are:

$$x_i = i\Delta x = ih \qquad \text{for} \qquad i = 0,1,2,3\ldots l$$

$$t_n = n\Delta t = nk \qquad \text{for} \qquad n = 0,1,2,3\ldots$$

At any interior mesh points (x_i, t_n), then the Equation of lateral heat loss (1) becomes

$$\frac{\partial u(x_i,t_n)}{\partial t} + c\,u(x_i,t_n) = a\frac{\partial^2 u(x_i,t_n)}{\partial x^2} \qquad (22)$$

The method is obtained using the central difference approximation for the first and second order partial derivatives with respect to t.

So that eq. (6) becomes

$$\frac{1}{2(\Delta t)}(u_i^{n+1} - u_i^{n-1}) - \frac{(\Delta t)^2}{6}\frac{\partial^2 u(x_i,u_n)}{\partial t^2} + cu_i^n$$

$$= \frac{1}{(\Delta t)^2}(u_{i+1}^n - 2u_i^n + 2u_{i-1}^n) - \frac{a(\Delta t)^2}{12}\frac{\partial^4 u(\xi_i\ ,t_n)}{\partial t^4}$$

where $\xi_i = (x_i, x_{i+1})$ and $\mu_n = (t^n, t^{n+1})$

Neglecting the truncation error leads to the difference equation.

$$\frac{1}{2(\Delta t)}(u_i^{n+1} - u_i^{n-1}) + cu_i^n = \frac{a}{(\Delta x)^2}$$

$$(u_{i+1}^n - 2u_i^n + u_{i-1}^n)$$

$$= \frac{a}{(\Delta t)^2}(u_{i+1}^n + u_{i-1}^n) = \left(\frac{1}{2(\Delta t)}\right)u_i^{n+1} +$$

$$\left(c + \frac{2a}{(\Delta t)^2}\right)u_i^n - \left(c + \frac{1}{2(\Delta t)}\right)u_i^{n-1}$$

Taking

$$\left(\frac{1}{2(\Delta t)}\right) = \lambda_1 \text{ and } \left(c + \frac{2a}{(\Delta t)^2}\right) = \lambda_2$$

So

$$\frac{a}{(\Delta t)^2}(u_{i+1}^n + u_{i-1}^n) = \lambda_1 u_i^{n+1} + \lambda_2 u_i^n - \lambda_1 u_i^{n-1}$$

$$\lambda_1 u_i^{n+1} = \frac{a}{(\Delta t)^2}(u_{i+1}^n + u_{i-1}^n) - \lambda_2 u_i^n + \lambda_1 u_i^{n-1}$$

$$u_i^{n+1} = \frac{a}{\lambda_1(\Delta t)^2}(u_{i+1}^n + u_{i-1}^n) - \frac{\lambda_2}{\lambda_1}u_i^n + u_i^{n-1}$$

By letting $\frac{a}{\lambda_1(\Delta x)^2} = n$, and $\frac{-\lambda_2}{\lambda_1} = \ \equiv$

So

$$u_i^{n+1} = \Pi\,(u_{i+1}^n + u_{i-1}^n) + \equiv u_i^n + u_i^{n-1}$$

$$u_i^{n+1} = \equiv u_i^n + \Pi u_{i+1}^n + \Pi u_{i-1}^n + u_i^{n-1} \qquad (23)$$

This equation holds for each $i = 1,2,\ldots,(l-1)$. The boundary conditions give

$$u_o^n = u_l^n = 0 \qquad (24)$$

for each n = 1, 2, ….

And the initial condition implies that

$$u_i^o = f(x_i) \qquad (25)$$

for $i = 1,2,\ldots,(l-1)$

Writing in matrix form for $i = 1,2,\ldots,(l-1)$ we have:

$$\begin{bmatrix} u_1^{n+1} \\ u_2^{n+1} \\ \vdots \\ \vdots \\ u_{l-1}^{n+1} \end{bmatrix} \begin{bmatrix} \equiv & \Pi & 0 & & 0 \\ \Pi & \equiv & \Pi & & \\ 0 & \Pi & \ddots & \ddots & 0 \\ & & \ddots & \equiv & \Pi \\ 0 & & 0 & \Pi & \equiv \end{bmatrix} \begin{bmatrix} u_1^n \\ u_2^n \\ \vdots \\ \vdots \\ u_{l-1}^n \end{bmatrix} + \begin{bmatrix} u_1^{n-1} \\ u_2^{n-1} \\ \vdots \\ \vdots \\ u_{l-1}^{n-1} \end{bmatrix} \qquad (26)$$

Equations (23) and (26) imply that the $(n+1)^{th}$ time steps requires values from the $(n)^{th}$ and $(n-1)^{th}$ time steps. This produces a minor starting problem since values of $n = 1$.

which are needed, in equation (26) to compute u_i^2 must be obtained from the initial value condition.

$$u\big|_i^o = f(x_i) \qquad 0 \le x \le l$$

A better approximation $u\big|_i^o$ can be obtained rather easily, particularly when the second derivative of $'f'$ at $'x_i'$ can be determined and it is already obtained in equation (21).

4. Stability Analysis

For the stability of the difference scheme (19), we assume that the solution of (19) at the grid point (*ih, nk*) is of the form

$$u_i^n = \eta^n e^{lkx} \qquad (27)$$

where $l = \sqrt{-1}$, where k is real and η is, in general complex. Substituting (27) into (19), we obtain a characteristic equation

$$\eta^2 - \frac{[2 \wedge coxX + \psi]}{[2\phi\, coxX + Y]}\eta - 1 = 0 \qquad (28)$$

where $X = k\,(\Delta x)$.

Equation (28) can be written as

$$\eta^2 - 2\rho\eta - 1 = 0 \qquad (29)$$

where $$\rho = \frac{[2 \wedge coxX + \psi]}{2[2\phi\, coxX + Y]} \qquad (30)$$

Equation (29) is a quadratic in η and hence will have two roots, i.e., $\eta = \rho \pm \sqrt{\rho^2 + 1}$. For stability, we must have $|\eta| \le 1$. The two roots can be written as

$$\eta_1 = \rho + \sqrt{\rho^2 + 1}$$

and

$$\eta_2 = \rho - \sqrt{\rho^2 + 1}$$

Now after substituting the values of $\wedge, \psi, \phi$ and Y, we have

$$\rho = \frac{-\left[\dfrac{c}{(12a^2)(12a + h^2 c)} + \dfrac{4}{h^2}\sin^2\dfrac{\chi}{2}\right]}{\left[\dfrac{c}{12ka^2}((12a + h^2 c) - \dfrac{1}{3ak}\sin^2\dfrac{\chi}{2}\right]}$$

If $\rho < 0$ then $\rho + \sqrt{\rho^2 + 1} < 1$ and $\rho - \sqrt{\rho^2 + 1} > -1$. So equation (19) is conditionally stable if $\rho < 0$.

5. Application 1

Consider the equation of lateral heat loss $u_{xx} = u_t + u$ in the interval $0 < x < \pi$. The boundary conditions are:

$$u(0,t) = u(\pi,t) = 0$$

and the initial conditions are

$$u(x,0) = \sin x, \;\; 0 \le x \le \pi, \qquad t > 0$$

The Exact Solution is $u(x,t) = e^{-2t} \sin x$.

Comparison of the Numerical Results of FDM and FOCM at t = 0.2

Table 1 for FDM

x_i	FDM	Exact	Error
0.000000000	0.00000000	0.00000000	0.00000000
0.314159265	0.21043594	0.20714029	0.00329565
0.628318531	0.40027294	0.39400424	0.00626870
0.942477796	0.55092847	0.54230031	0.00862816
1.256637061	0.64765527	0.63751225	0.01014302
1.570796327	0.68098505	0.67032005	0.01066500
1.884955592	0.64765527	0.63751225	0.01014302
2.199114858	0.55092847	0.54230031	0.00862816
2.513274123	0.40027294	0.39400424	0.00626870
2.827433388	0.21043594	0.20714029	0.00329565
3.141592654	0.00000000	0.00000000	0.00000000

Table 2 for FOCM

x_i	FOCM	Exact	Error
0.000000000	0.00000000	0.00000000	0.00000000
0.314159265	0.21003558	0.20714029	0.00289529
0.628318531	0.39951135	0.39400424	0.00550711
0.942477796	0.54988026	0.54230031	0.00757995
1.256637061	0.64642299	0.63751225	0.00891074
1.570796327	0.67968936	0.67032005	0.00936931
1.884955592	0.64642299	0.63751225	0.00891074
2.199114858	0.54988026	0.54230031	0.00757995
2.513274123	0.39951135	0.39400424	0.00550711
2.827433388	0.21003558	0.20714029	0.00289529
3.141592654	0.00000000	0.00000000	0.00000000

For graph see Figure 1

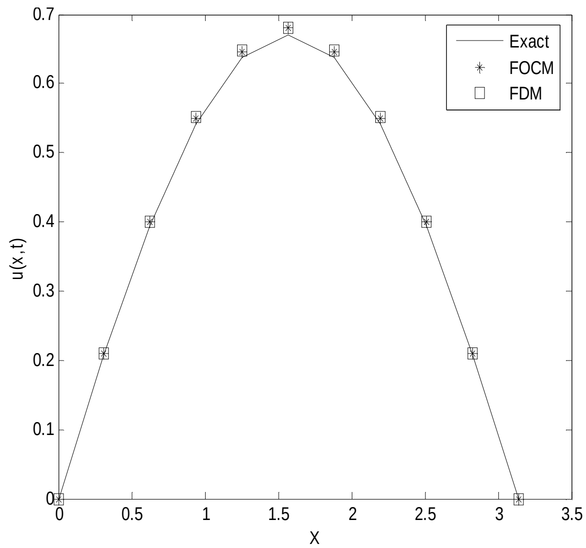

Fig. 1 Comparison of FDM, FOCM, and exact value

6. Application 2

Consider the equation of lateral heat loss $3u_{xx} = u_t + u$ in the interval $0, x < \pi$. The boundary conditions are:

$$u(0,t) = u(\pi,t) = 0$$

and the initial conditions are

$$u(x,0) = \sin x, \;\; 0 \le x \le \pi, \qquad t > 0$$

The Exact Solution is $u(x,t) = e^{-4t} \sin x$

Comparison of the Numerical Results of FDM and FOCM at t = 0.2

Table 3 for FDM

x_i	FDM	Exact	Error
0.000000000	0.00000000	0.00000000	0.00000000
0.314159265	0.16099658	0.13885029	0.02214629
0.628318531	0.30623365	0.26410894	0.04212471
0.942477796	0.42149446	0.36351477	0.05797969
1.256637061	0.49549649	0.42733724	0.06815925
1.570796327	0.52099578	0.44932896	0.07166682
1.884955592	0.49549649	0.42733724	0.06815925
2.199114858	0.42149446	0.36351477	0.05797969
2.513274123	0.30623365	0.26410894	0.04212471
2.827433388	0.16099658	0.13885029	0.02214629
3.141592654	0.00000000	0.00000000	0.00000000

Table 4 for FOCM

x_i	FOCM	Exact	Error
0.000000000	0.00000000	0.00000000	0.00000000
0.314159265	0.15356582	0.13885029	0.01471553
0.628318531	0.29209955	0.26410894	0.02799061
0.942477796	0.40204054	0.36351477	0.03852577
1.256637061	0.47262701	0.42733724	0.04528977
1.570796327	0.49694944	0.44932896	0.04762048
1.884955592	0.47262701	0.42733724	0.04528977
2.199114858	0.40204054	0.36351477	0.03852577
2.513274123	0.29209955	0.26410894	0.02799061
2.827433388	0.15356582	0.13885029	0.01471553
3.141592654	0.00000000	0.00000000	0.00000000

For graph see Figure 2

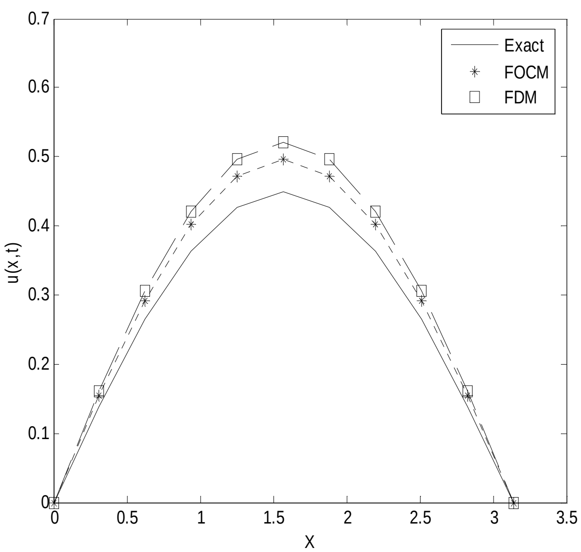

Fig. 2 Comparison of FDM, FOCM and EXACT value.

7. Results and Discussion

In this paper, numerical solutions of the one dimensional linear equation of lateral heat loss are derived using Finite Difference Method (FDM) and ZZ Fourth Order Compact Method (FOCM). ZZ Fourth Order Compact Method is known to be a powerful device for solving functional equations. Using Von Neumann Stability, it is found that the system derived for Lateral Heat loss is conditionally stable. From the solutions of the equation of lateral heat loss, we note that the fourth order compact method, gives better results than the usual second order method.

8 References

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