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Some Finite Difference Methods for One Dimensional Burger's Equation for Irrotational Incompressible Flow Problem

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Abstract

In this paper special case of famous Burgers' Equation in one dimension is solved numerically by three approaches which are FTCS explicit scheme, BTCS implicit scheme and Mac-Cormack explicit scheme. The equation in itself is important to understand the concept of fluid flow with constant pressures and irrotational flows. Four cases are discussed here. These are for four different times i.e., T = 0.1, 0.4, 0.7, 1.0. Results for these cases analyzed and shown in the form of plots. Numerical results show that the BTCS implicit method seems more accurate over the others for the cases discussed.

Key Words: Burger's Equation; BTCS Method; FTCS Method; Mac-Cormack Method

1. Introduction

Burgers' equation being a simplified form of the Navier-Stokes' equation and having immense physical applications has attracted researcher's attention since the past few decades [1].For over sixty years, Burger's equation [2, 3] has been studied and used as a simple model for many physically interesting problems and for convection–diffusion phenomena such as shock waves, turbulence, decaying free turbulence, traffic flows, flow related problems, etc. Burger's equation can be thought of as a one dimensional analog of the Navier-Stokes equations which model the behavior of viscous fluids. As such, it is a useful model equation on which to investigate techniques that might be applied to complicated fluid flow problems [4].Three finite difference schemes FTCS, BTCS and Mac-Cormack [5] Method have been used to solve special case of one dimensional Burger's equation with unit viscosity. Numerical results are analyzed and compared with exact solutions.

2. Forward Time Centered Space (FTCS) Method

The non-linear second order, Burger's Equation with unit viscosity is:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \qquad (1)$$

With boundary condition

$$u(-9.0,t)=2.0, \quad u(9.0,t)=-2.0 \qquad t>0$$

and initial condition

$$u(x,0.1) = \frac{-2\, Sin\, hx}{Cos\, hx - e^{-0.1}}$$

Using FTCS Method, we can write (1) as

$$\frac{u_{i,j+1}-u_{i,j}}{k} + u_{i,j}\,\frac{u_{i+1,j}-u_{i-1,j}}{2h} = \frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}$$

$$u_{i,j+1} = u_{i,j} - \frac{k}{2h}u_{i,j}\left(u_{i+1,j} - u_{i-1,j}\right) + \frac{k}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

$$u_{i,j+1} = u_{i,j} - c\,u_{i,j}\left(u_{i+1,j} - u_{i-1,j}\right) + d\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

Where $c = \frac{k}{2h}$ and $d = \frac{k}{h^2}$

$$u_{i,j+1} = (1-2d)u_{i,j} + (d + c u_{i,j})u_{i-1,j} + (d - cu_{i,j})\,u_{i+1,j}$$

$$u_{i,j+1} = (d + c\,w_{i,j})u_{i-1,j} + (1-2d)u_{i,j} + (d - cu_{i,j})\,u_{i+1,j}$$

$$u_{i,j+1} = (d + c\, w_{i,j})\, u_{i-1,j} + (1-2d)\, u_{i,j} + (d - c\, u_{i,j})\, u_{i+1,j}$$

Using above scheme, solution at T = 0.1, 0.4, 0.7 and 1.0 is approximated with h = 0.2 and k = 0.01. The results are presented in Figure 1.

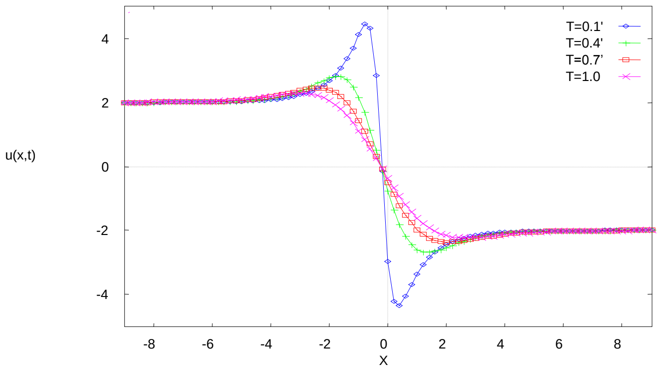

Figure 1: Burgers Equation FTCS h = 0.2 k = 0.01 T = 0.1,0.4,0.7 AND 1.0

3. Backward Time Centered Space (BTCS) Method

Using BTCS Method, we can write (1) as

$$\frac{u_{i,j+1} - u_{i,j}}{k} + u_{i,j}\, \frac{u_{i+1,j+1} - u_{i-1,j+1}}{2h} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}$$

$$u_{i,j+1} - u_{i,j} + \left(\frac{k}{2h}\right) u_{i,j} \left(u_{i+1,j+1} - u_{i-1,j+1}\right) = \left(\frac{k}{h^2}\right)\left(u_{i+1,j+1} - 2u_{i-1\,j+1} + u_{i-1},_{j+1}\right)$$

$$-\left(\frac{k}{h^2} + \frac{k}{2h} u_{i,j}\right) u_{i-1,j+1}, + \left(1 + 2\frac{k}{h^2}\right) u_{i,j+1} + \left(\frac{k}{h^2} u_{i,j} - \frac{k}{h^2}\right) u_{i+1,j+1} = u_{i,j}$$

$$-\left(d + c\, u_{i,j}\right) u_{i-1,j+1} + \left(1 + 2d\right) u_{i,j+1} + \left(c\, u_{i,j} - d\right) u_{i+1,j+1} = u_{i,j}$$

Where $c = \dfrac{k}{2h}$ and $d = \dfrac{k}{h^2}$

Solution at T = 0.1, 0.4, 0.7 and 1.0 is approximated using the BTCS method with h = 0.2 and k = 0.01. The results are presented in the Figure 2.

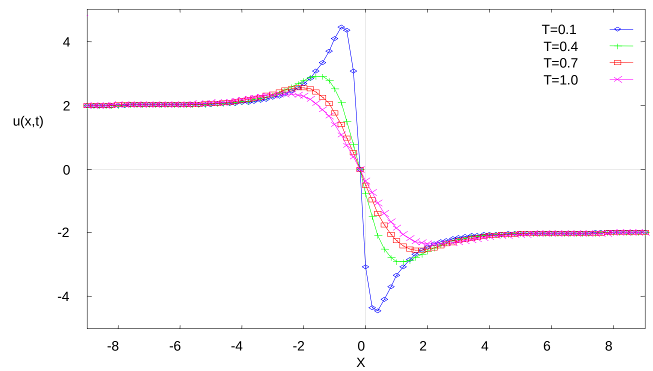

Figure 2: Burgers Equation BTCS h = 0.2 k = 0.01 T = 0.1,0.4,0.7 AND 1.0

4. Mac-Cormack Method

Let us consider the non-linear second order, Burger's Equation with unit viscosity,

$$\frac{\partial x}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2} \qquad (2)$$

Or $u_t + u\, u_x = u_{xx}$

$$u_t = -u\, u_x + u_{xx} \qquad (3)$$

4.1 Predictor

We can write (2) as

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} + u_{i,j}\, \frac{u_{i+1,j} - u_{i,j}}{\Delta x} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2}$$

$$\Rightarrow u_{i,j+1} = u_{i,j} - \frac{\Delta t}{\Delta x}\left(u_{i+1,j} - u_{i,j}\right) u_{i,j} + \frac{\Delta t}{(\Delta x)^2}\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right)$$

$$u_{i',j+1} = u_{i,j} - c\left(u_{i+1,j} - u_{i,j}\right)u_{i,j} + d\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right) \quad (4)$$

Where $c = \frac{\Delta t}{\Delta x}$ & $d = \frac{\Delta t}{(\Delta x)^2}$

$$u_{i,p} = d\,u_{i-1,j} + (1 + c u_{i,j} - 2d)u_{i,j} + (d - c)u_{i+1,j} \quad (5)$$

4.2 Corrector

Expanding u(x, t) and u_t (x, t) by Taylor's series in time about grid point (i, j)

$$u_{i,j+1} = u_{i,j} + u_t|_i^j \Delta t + \frac{1}{2}u_{tt}|_i^j \Delta t^2 + O(\Delta t^3) \quad (6)$$

and $$u_t|_i^{j+1} = u_t|_i^j + u_{tt}|_i^j \Delta t + O(\Delta t^2)$$

$$\Rightarrow \quad u_{tt}|_i^j = \frac{u_t|_i^{j+1} - u_t|_i^j}{\Delta t}$$

∴ Equation (6) can be written as

$$u_{i,j+1} = u_{i,j} + u_t|_i^j \Delta t + \frac{1}{2}\left(\frac{u_t|_i^{j+1} - u_t|_i^j}{\Delta t}\right)\Delta t^2$$

$$u_{i,j+1} = u_{i,j} + \frac{1}{2}\left(u_t|_i^{j+1} - u_t|_i^j\right)\Delta t$$

Using equation (3), we have

$$u_{i,j+1} =$$

$$u_{i,j} + \frac{1}{2}\left[u u_x|_i^{j+1} + u_{xx}|_i^{j+1} - u u_x|_i^j + u_{xx}|_i^j\right]\Delta t$$

$$= u_{i,j} + \frac{1}{2}[-u_{i,\overline{j+1}}\frac{u_{i,\overline{j+1}} - u_{i-1,\overline{j+1}}}{\Delta x} + \frac{u_{i-1,\overline{j+1}} - 2u_{i,\overline{j+1}} + u_{i+1,\overline{j+1}}}{(\Delta x)^2}$$

$$- u_{i,j}\frac{(u_{i+1,j} - u_{i,j})}{\Delta x} + \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{(\Delta x)^2}]\Delta t$$

$$u_{i,j+1} = u_{i,j} + \frac{1}{2}\Big[-c\left(u_{i,\overline{j+1}} - u_{i-1,\overline{j+1}}\right)u_{i,\overline{j+1}} + d\left(u_{i-1,\overline{j+1}} - 2u_{i,\overline{j+1}} + u_{i+1,\overline{j+1}}\right)$$

$$- c\left(u_{i+1,j} - u_{i,j}\right)u_{i,j} + d\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right)\Big]$$

Where $c = \frac{\Delta t}{\Delta x}$ & $d = \frac{\Delta t}{(\Delta x)^2}$

Using equation (4) , we have

$$u_{i,j+1} = u_{i,j} + \frac{1}{2}\Big[-c\left(u_{i,\overline{j+1}} - u_{i-1,\overline{j+1}}\right)u_{i,\overline{j+1}} + d\left(u_{i-1,\overline{j+1}} - 2u_{i,\overline{j+1}} + u_{i+1,\overline{j+1}}\right) + u_{i,j+1} - u_{i,j}\Big]$$

Now

$$u_{i,c} = \frac{1}{2}u_{i,j} + \frac{1}{2}u_{i,\overline{j+1}} + \frac{1}{2}\left[d\,u_{i-1,\overline{j+1}} - \left\{c\left(u_{i,\overline{j+1}} - u_{i-1,\overline{j+1}}\right) + 2d\right\}u_{i,\overline{j+1}} + d\,u_{i+1,\overline{j+1}}\right]$$

$$= \frac{1}{2}\left[u_{i,j} + u_{i,p} + d\,u_{i-1,p} - \left\{c\left(u_{i,p} - u_{i-1,p}\right) + 2d\right\}u_{i,p} + d\,u_{i+1,p}\right]$$

$$= \frac{1}{2}\Big[u_{i,j} + d\,u_{i-1,p} + \left\{1 - c\left(u_{i,p} - u_{i-1,p}\right) - 2d\right\}u_{i,p} + d\,u_{i+1,p}\Big]$$

$$u_{i,c} = \frac{1}{2}\Big[u_{i,j} + d\,u_{i-1,p} + \left\{1 - c\left(u_{i,p} - u_{i-1,p}\right) - 2d\right\}u_{i,p} + d\,u_{i+1,p}\Big]$$

Solution at T = 0.1, 0.4, 0.7 and 1.0 is approximated using the Mac-Cormack method with h = 0.2 and 0.01. The results are presented in the Figure3.

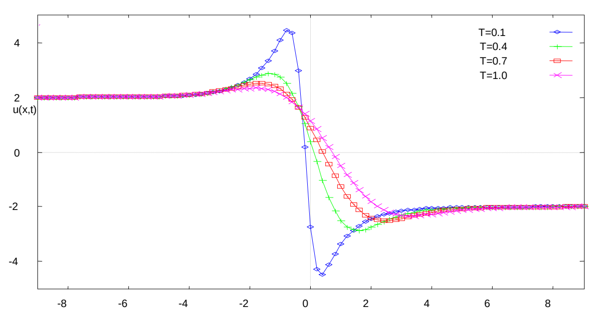

Figure 3: Burger's Equation MACCORMACK
h = 0.2 k = 0.01 T = 0.1,0.4,0.7 and 1.0

5. Results and Discussion

The results are shown in figures (1-4) as plots for various times for all the three methods. Figure 4 shows the comparison of all the methods for time 0.4 sec. It seems that all the methods behave almost similarly and deviate from the exact solution from –2 to 2. Figure 1shows the FTCS explicit method for all the times i.e., T = 0.1, 0.4, 0.7, 1.0. Similarly Figure 2 and 3 show the BTCS implicit method and the Mac-Cormack method for all the times. It seems that the BTCS implicit method is giving more accurate results for time 0.7 and 1.0. However at the time 0.1 all methods behave in a similar manner and deviate a lot for x lies between –0.5 to 0.5. At time 0.4 second Mac-Cormack method seems to be more accurate.

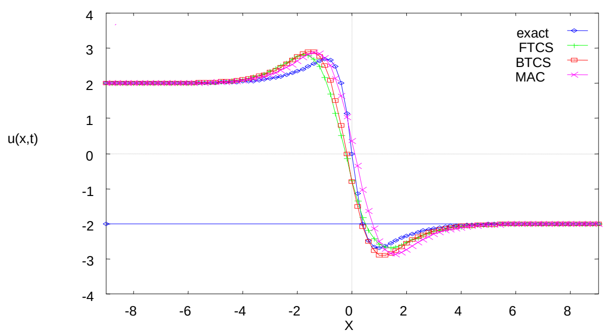

Figure 4: Burgers Equation EXACT, FTCS, BTCS AND MACC h = 0.2 k = 0.01 T = 0.4

6. Conclusion

It can be concluded on bases of these results that BTCS implicit method is more accurate for times 0.7 & 1.0 seconds and the Mac-Cormack is better for time 0.4 sec. However all the methods behave in a similar manner for the values from –2.0 to 2.0 and deviate a lot.

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