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Pythagorean Fuzzy Multisets and their Applications to Therapeutic Analysis and Pattern Recognition

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Abstract. Most of the real life problems embroil uncertainties, imprecision and vagueness. Fuzzy multisets and Pythagorean fuzzy sets, initially suggested by Yager, are significant mathematical models to handle such real world problems. By combining these two notions, we introduce a new kind of hybrid mathematical model: Pythagorean fuzzy multisets (PFM-sets). We present some prime concepts of Pythagorean fuzzy multisets and establish various algebraic operations on them along with some important results. We render two applications to multi-attribute group decision making (MAGDM), accompanied by algorithms and flow charts, established on PFM-sets: One in therapeutic analysis linking medical and mathematical sciences and the other in pattern recognition.

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1. Introduction

The ground stone of contemporary mathematics is believed to be founded upon two mainstays: mathematical rationality and set philosophy. These two pillars undeniably compose the language bridging in more or less all fields of mathematics. In fact, the prompt advancement of science has directed towards an imperative requirement for the growth of current set theoretic mathematical and carefully worked-out modeling. A (crisp) set is always concomitant with a characteristic function. Taking into account the uncertainty element, Zadeh [62] in 1965, suggested fuzzy sets in which a membership function is allocated to each affiliate of the universe of discourse. Following the footsteps of Zadeh, numerous theories and approaches treating uncertainty, imprecision and vagueness have been proposed so far.
In 1983, Atanassov [6, 7] introduced intuitionistic fuzzy sets (IF-sets) bearing membership and non-membership functions. Atanassov also familiarized geometrical elucidation of the elements of intuitionistic fuzzy objects [8] and introduced intuitionistic fuzzy multi-dimensional sets and intuitionistic fuzzy negations [9]-[12]. Amongst a number of higher order fuzzy models, intuitionistic fuzzy sets arrange for a supple structure to elaborate imprecision. IFS reflects better the ever-changing trait of human manners and attitude of behaving about things and situations at different situations. A person who articulates the level of belongingness of some given element to a set, usually does not put across its counter-part viz. level of non-belongingness. This psychosomatic fact describes that negating linguistically does not always happen together with logical negation. The name intuitionistic fuzzy set is owing to George Gargove, with the inspiration that their fuzzification negates the renowned law of excluded middle which is one of the principal accepted wisdom of *intuitionism* – a philosophy of mathematics that was introduced by the Dutch mathematician Brouwer (1881-1966). It is founded upon the notion that mathematics is a conception of the brain. The accuracy of a mathematical statement can simply be conceived via a mental structure that shows it to be honest, and the communication between mathematicians only serves as a way to produce the same mental operation in different brains. The notion of intuitionism looks like expedient in modeling many real life state of affairs including psychological investigations, logical reasoning, and negotiation processes etc.
Many areas of modern mathematics have been merged by encroaching upon a rudimentary standard of a given theory only because advantageous structures could be set this way. Contrary to ordinary sets, multisets permit us to have multiple occurrences of the members. Blizard [13, 14] introduced multiset theory as a generalization of crisp set theory. As a broad view of multiset, Yager [58] introduced the

notion of fuzzy multiset (FMS). Pythagorean fuzzy set (PF-set), also known as IF-set of type-2 [8], introduced by Yager [59]-[61] is the extension of intuitionistic fuzzy set (IF-set) introduced by Atanassov [6]. Yager also presented Pythagorean membership grades with applications to the multi-criteria decision making (MCDM). Peng *et al.* [36], and Guleria and Bajaj [25] presented Pythagorean fuzzy soft sets (PFS-sets). Some of the prominent researchers who presented different hybrid structures of fuzzy sets and their applications include Akram *et al.* [1]-[3], Ali *et al.* [4, 5], Davvaz and Sadrabadi [15], Ejegwa and Modom [17], Rajarajeswari and Uma [38], Feng *et al.* [18]-[20], Garg [21]-[24], Kumar and Garg [26], Karaaslan [27, 28], Naeem *et al.* [31, 32], Peng *et al.* [33]-[37], Riaz *et al.* [39]-[45], Riaz and Hashmi [46, 47], Riaz and Tehrim [48]-[50], Shinoj and John [51, 52], Tehrim and Riaz [53], Wei [54], Xu [55], Xu *et al.* [56, 57], Zhang and Xu [65], and Zhan *et al.* [66]-[68]. Zararsiz [63] discussed similarity measures of sequence of fuzzy numbers and fuzzy risk analysis presenting fabulous application using center of gravity points. Zararsiz [64] calculated entropy and similarity measure values by using the amplitude and the duration of QRS-complexes of sport horse and gave some valuable comments.
Malik and Riaz [29]-[30] introduced G-subsets and G-orbits of $Q^*(\sqrt{n})$ under action of the Modular Group. Zhang and Zhan [69] introduced novel classes of fuzzy soft β-covering based fuzzy rough sets with applications to multi-criteria fuzzy group decision making.

The goal of this article is to introduce Pythagorean fuzzy multisets that has natural applications in multiple-valued logic, multi-sensor, multi-source and multi-process information fusion. Pythagorean fuzzy multisets provide a strong mathematical model to take in hand multi-attribute group decision making (MAGDM). While tackling real world problems, intuitionistic fuzzy multiset cannot deal with the situation if the sum of membership degree and non-membership degree of the parameter gets larger than 1. It makes decision making demarcated, and affects the optimum decision. PFM-sets assist us in handling such situations. PFM-sets provide a large number of applications to MAGDM problems in artificial intelligence, image processing, medical diagnosis, forecasting, recruitment problems and many other real life problems.

The article is sorted out as pursues. In Section 2, we bring to mind some rudiments of fuzzy set, multiset, fuzzy multiset, intuitionistic fuzzy set, intuitionistic fuzzy multiset and Pythagorean fuzzy set. We introduce, in Section 3, PFM-set and its properties with the assistance of examples. In Section 4, we propose Algorithm 1 for multi-attribute group decision making (MAGDM) based on PFM-sets to medical diagnosis which correlates once thought pole apart fields Mathematics and Medical science. In Section 5, we propose Algorithm 2 for MAGDM to pattern recognition. Furthermore, we explain the procedural steps of Algorithm 1 & Algorithm 2 with the assistance of flow charts. The effectiveness of proposed methods are also justified by the numerical examples. The idea of linkage of MAGDM with PFM-sets may be efficiently employed in diverse sectors of real life problems. We present the superiority of our proposed model over the prevailing models in section 6. Finally, in Section 7, a brief conclusion is presented.

2. Preliminaries

In this subdivision, we concisely call to mind some primary notions of different kinds of sets that would be employed in the remnant part of this article.

Definition 2.1. [62] Presume that X is a non-empty set. A *fuzzy set* (an FS for short) defined over a non-void collection X comprises ordered pairs in which abscissa is from X and the ordinate is a mapping ζ (acknowledged as membership function) that drives elements of X to the interval $[0,1]$.

Definition 2.2. [13, 14] A *multiset* (mset for short) over X is a couple $< X, d >$, where $d : X \mapsto \mathbb{N}$ is a function over an underlying crisp set X. A multiset M is given by

$$\begin{aligned} M &= < X, d > \\ &= \Big[\frac{d(\varrho_i)}{\varrho_i} : i = 1,2,3,\cdots,n\Big] \end{aligned}$$

where $d(\varrho_i)$ is the duplicity of $\varrho_i \in X$. For example, if $X = \{\varrho_i : i = 1,2,3\}$ then

$$\begin{aligned} M &= \{\varrho_1, \varrho_1, \varrho_2, \varrho_3, \varrho_3, \varrho_3, \varrho_3\} \\ &= \Big[\frac{2}{\varrho_1}, \frac{1}{\varrho_2}, \frac{4}{\varrho_3}\Big] \end{aligned}$$

is a multiset over X.
Other synonyms used in literature for mset are *bag*, *list*, *bunch*, *heap*, *sample*, *weighted set*, *occurrence set*, and *fireset*.

Definition 2.3. [58] Let X be a non-empty set. A *fuzzy multiset* (an FMS for short) A is typified by a mapping $\alpha_A : X \mapsto M$ (conventionally called count membership), where M is the collection of all multisets extracted from the unit closed interval. For every $\varrho \in X$, the affiliation sequence (usually called membership sequence) is defined as decreasingly organized sequence of elements in $\alpha_A(\varrho)$ and is characterized as $\big(\zeta_A^{(1)}(\varrho), \zeta_A^{(2)}(\varrho), \cdots, \zeta_A^{(n)}(\varrho)\big)$ with the constraint that $\zeta_A^{(i)}(\varrho) \geq \zeta_A^{(i+1)}(\varrho)$.

Definition 2.4. [7] An *intuitionistic fuzzy set* (IFS in brief) over the underlying non-empty set X is expressed as

$$A = \{< \varrho, \zeta_A(\varrho), \xi_A(\varrho) >: \varrho \in X\}.$$

The mappings ζ_A and ξ_A in order are acknowledged as the degrees of membership and non-membership of the element $\varrho \in X$ to the set A and drag elements of X to unit closed interval along with the constraint that their sum must not exceed unity. It is pertinent to notice that every FS A may be thought of an IFS of the form

$$A = \{< \varrho, \zeta_A(\varrho), 1 - \zeta_A(\varrho) >: \varrho \in X\}.$$

Definition 2.5. [51] An *intuitionistic fuzzy multiset* (an IFMS for short) A over the underlying non-empty set X is portrayed by two mappings $\zeta_A : X \mapsto M$ (conventionally called membership count) and $\xi_A : X \mapsto M$ (conventionally called non-membership count), where M is the collection of all multisets extracted from the unit closed interval. For every $\varrho \in X$, the membership sequence is defined as decreasingly arrayed progression of elements in $\zeta_A(\varrho)$ represented as

$$\big(\zeta_A^{(1)}(\varrho), \zeta_A^{(2)}(\varrho), \cdots, \zeta_A^{(n)}(\varrho)\big)$$

bearing the constraint $\zeta_A^{(i)}(\varrho) \geq \zeta_A^{(i+1)}(\varrho)$. The corresponding non-membership sequence is represented as $\big(\xi_A^{(1)}(\varrho), \xi_A^{(2)}(\varrho), \cdots, \xi_A^{(n)}(\varrho)\big)$. Further, $0 \leq \zeta_A^{(i)}(\varrho) + \xi_A^{(i)}(\varrho) \leq 1$, for all i.
An IFMS may be expressed in set-builder notation as

$$A = \big\{ < \varrho : \big(\zeta_A^{(1)}(\varrho), \zeta_A^{(2)}(\varrho), \cdots, \zeta_A^{(n)}(\varrho)\big), \big(\xi_A^{(1)}(\varrho), \xi_A^{(2)}(\varrho), \cdots, \xi_A^{(n)}(\varrho)\big) >: \varrho \in X \big\}.$$

It is remarkable to note that the non-membership sequence need not to be in ascending or descending order, contrary to the membership sequence.

Definition 2.6. [33] A *Pythagorean fuzzy set*, abbreviated as PF-set, is a family of the form

$$P = \big\{ < \varrho, \zeta_P(\varrho), \xi_P(\varrho) >: \varrho \in X \big\}$$

where ζ_P and ξ_P are mappings from some crisp set X to the unit interval with the restriction that sum of their squares should not exceed unity i.e. $0 \leq \zeta_P^2(\varrho) + \xi_P^2(\varrho) \leq 1$, called correspondingly the grade of association and non-association of $\varrho \in X$ to the set P. The doublet (ζ_p, ξ_p) is called Pythagorean Fuzzy Number, abbreviated as PFN. The space for a PF-set is a unit circular arc in the first quadrant whereas it was a right isosceles triangle having length of the base and altitude each equal to unity in case of an if-set. Hence we have an enlarged space for PF-sets as compared to IF-sets. Indeed we have included $\frac{2\pi-1}{8}$ extra area for PF-sets.

3. Pythagorean Fuzzy Multisets

In this section, we present the notion of PFM-sets followed by some of their algebraic properties. We shall use $\zeta_{P_j}^{(i)}$ to mean $\zeta_{P_j}^{(i)}(\varrho)$, for $\varrho \in X$, just for the sake of brevity. The same is the situation for $\xi_{P_j}^{(i)}$.

Definition 3.1. A *Pythagorean fuzzy multiset* (a PFM-set for short) P over a non-empty underlying set X is characterized by two mappings $\zeta_P : X \mapsto M$ (traditionally acknowledged membership count) and $\xi_P : X \mapsto M$ (conventionally called non-membership count), where M is the collection of all multisets drawn from the unit closed interval. For every $\varrho \in X$, the membership sequence is defined as descending ordered progression of members in $\zeta_P(\varrho)$ represented as $\big(\zeta_P^{(1)}(\varrho), \zeta_P^{(2)}(\varrho), \cdots, \zeta_P^{(n)}(\varrho)\big)$ where $\zeta_P^{(i)}(\varrho) \geq \zeta_P^{(i+1)}(\varrho)$. The corresponding non-membership sequence is represented as $\big(\xi_P^{(1)}(\varrho), \xi_P^{(2)}(\varrho), \cdots, \xi_P^{(n)}(\varrho)\big)$. Further, $0 \leq \Big(\zeta_P^{(i)}(\varrho)\Big)^2 + \Big(\xi_A^{(i)}(\varrho)\Big)^2 \leq 1$, for all i.
A PFM-set may be expressed in set-builder notation as

$$\begin{aligned} P &= \big\{ < \varrho : \big(\zeta_P^{(1)}(\varrho), \zeta_P^{(2)}(\varrho), \cdots, \zeta_P^{(n)}(\varrho)\big), \big(\xi_P^{(1)}(\varrho), \xi_P^{(2)}(\varrho), \cdots, \xi_P^{(n)}(\varrho)\big) >: \varrho \in X \big\} \\ &= \big\{ < \varrho : \big(\{\zeta_P^{(i)}(\varrho)\}_{i=1}^n\big), \big(\{\xi_P^{(i)}(\varrho)\}_{i=1}^n\big) >: \varrho \in X \big\} \end{aligned}$$

or more conveniently as

$$P = \left\{ \frac{\varrho}{\left(\left\{\zeta_P^{(i)}(\varrho)\right\}_{i=1}^{n}\right), \left(\left\{\xi_P^{(i)}(\varrho)\right\}_{i=1}^{n}\right)} : \varrho \in X \right\}.$$

It is remarkable to note that the non-membership sequence need not to be in ascending or descending order, contrary to the membership sequence.

Example 3.2. Let $X = \{\varrho_1, \varrho_2, \varrho_3\}$ be a crisp set, then

$$P = \left\{ \frac{\varrho_1}{(0.27, 0.12), (0.88, 0.56)}, \frac{\varrho_2}{(0.53, 0.24, 0.11), (0.62, 0.56, 0.67)}, \frac{\varrho_3}{(0.91, 0.57), (0.17, 0.67)} \right\}$$

is a PFM-set.

Definition 3.3. The cardinality of $\zeta_P(\varrho)$ (or that of $\xi_P(\varrho)$) in a PFM-set P is called *length of the element* $\varrho \in P$ and is designated as $L(\varrho : P)$ i.e.

$$L(\varrho : P) = \#\big(\zeta_P(\varrho)\big) = \#\big(\xi_P(\varrho)\big)$$

where $\#\big(\zeta_P(\varrho)\big)$ denotes the cardinality of the membership sequence $\zeta_P(\varrho)$ and $\#\big(\xi_P(\varrho)\big)$ that of non-membership sequence $\xi_P(\varrho)$.
For example, in Example 3.2

$$\begin{aligned} L(\varrho_1 : P) &= 2, \\ L(\varrho_2 : P) &= 3, \text{and} \\ L(\varrho_3 : P) &= 2. \end{aligned}$$

If P_1 and P_2 are two PFM-sets extracted from X, then

$$L(\varrho : P_1, P_2) = \max\big\{L(\varrho : P_1), L(\varrho : P_2)\big\}.$$

For the sake of transience, we use $L(\varrho)$ to mean $L(\varrho : P_1, P_2)$.

Definition 3.4. Two PFM-sets P_1 and P_2 drawn from a non-empty set X are said to be *equivalent*, written $P_1 \sim P_2$, if and only if $L(\varrho, P_1) = L(\varrho, P_2)$.

Definition 3.5. Let X be a crisp set having

$$P_1 = \left\{ \frac{\varrho}{\left(\left\{\zeta_{P_1}^{(i)}(\varrho)\right\}_{i=1}^{n}\right), \left(\left\{\xi_{P_1}^{(i)}(\varrho)\right\}_{i=1}^{n}\right)} : \varrho \in X \right\}$$

and

$$P_2 = \left\{ \frac{\varrho}{\left(\left\{\zeta_{P_2}^{(i)}(\varrho)\right\}_{i=1}^{n}\right), \left(\left\{\xi_{P_2}^{(i)}(\varrho)\right\}_{i=1}^{n}\right)} : \varrho \in X \right\}$$

as two PFM-sets. We say that P_1 is a *PFM-subset* of P_2, written $P_1 \sqsubseteq P_2$, if for all admissible values of i we have $\zeta_{P_1}^{(i)}(\varrho) \leq \zeta_{P_2}^{(i)}(\varrho)$ and $\xi_{P_1}^{(i)}(\varrho) \geq \xi_{P_2}^{(i)}(\varrho)$. If there is at least one i for which $\zeta_{P_1}^{(i)}(\varrho) < \zeta_{P_2}^{(i)}(\varrho)$ or $\xi_{P_1}^{(i)}(\varrho) > \xi_{P_2}^{(i)}(\varrho)$, then P_1 is called a *proper PFM-subset* of P_2, written $P_1 \sqsubset P_2$. P_1 and P_2 are said to be *equal* i.e. $P_1 = P_2$ if $P_1 \sqsubseteq P_2 \sqsubseteq P_1$.

Example 3.6. Let

$$P_1 = \left\{ \frac{\varrho_1}{(0.22, 0.21, 0.17), (0.13, 0.56, 0.81)}, \frac{\varrho_2}{(0.51, 0.34, 0.29), (0.62, 0.76, 0.69)}, \frac{\varrho_3}{(0.98, 0.57), (0.11, 0.37)} \right\}$$

and

$$P_2 = \left\{ \frac{\varrho_1}{(0.27, 0.21, 0.20), (0.01, 0.40, 0.42)}, \frac{\varrho_2}{(0.73, 0.39, 0.38), (0.62, 0.55, 0.34)}, \frac{\varrho_3}{(1, 0.74), (0, 0.29)} \right\}$$

be PFM-sets over $X = \{\varrho_1, \varrho_2, \varrho_3\}$, then $P_1 \sqsubseteq P_2$.

Definition 3.7. The *union* of two PFM-sets

$$P_1 = \left\{ \frac{\varrho}{\left(\{\zeta^{(i)}_{P_1}(\varrho)\}^n_{i=1}\right), \left(\{\xi^{(i)}_{P_1}(\varrho)\}^n_{i=1}\right)} : \varrho \in X \right\}$$

and

$$P_2 = \left\{ \frac{\varrho}{\left(\{\zeta^{(i)}_{P_2}(\varrho)\}^n_{i=1}\right), \left(\{\xi^{(i)}_{P_2}(\varrho)\}^n_{i=1}\right)} : \varrho \in X \right\}$$

drawn from X is defined as

$$P_1 \sqcup P_2 = \left\{ \frac{\varrho}{\left(\{\zeta^{(i)}_{P}(\varrho)\}^n_{i=1}\right), \left(\{\xi^{(i)}_{P}(\varrho)\}^n_{i=1}\right)} : \varrho \in X \right\}$$

where

$$\zeta^{(i)}_P(\varrho) = \max\left\{\zeta^{(i)}_{P_1}(\varrho), \zeta^{(i)}_{P_2}(\varrho)\right\}$$

and

$$\xi^{(i)}_P(\varrho) = \min\left\{\xi^{(i)}_{P_1}(\varrho), \xi^{(i)}_{P_2}(\varrho)\right\}.$$

Definition 3.8. The *intersection* of two PFM-sets

$$P_1 = \left\{ \frac{\varrho}{\left(\{\zeta^{(i)}_{P_1}(\varrho)\}^n_{i=1}\right), \left(\{\xi^{(i)}_{P_1}(\varrho)\}^n_{i=1}\right)} : \varrho \in X \right\}$$

and

$$P_2 = \left\{ \frac{\varrho}{\left(\{\zeta^{(i)}_{P_2}(\varrho)\}^n_{i=1}\right), \left(\{\xi^{(i)}_{P_2}(\varrho)\}^n_{i=1}\right)} : \varrho \in X \right\}$$

drawn from X is defined as

$$P_1 \sqcap P_2 = \left\{ \frac{\varrho}{\left(\{\zeta^{(i)}_{P}(\varrho)\}^n_{i=1}\right), \left(\{\xi^{(i)}_{P}(\varrho)\}^n_{i=1}\right)} : \varrho \in X \right\}$$

where

$$\zeta^{(i)}_P(\varrho) = \min\left\{\zeta^{(i)}_{P_1}(\varrho), \zeta^{(i)}_{P_2}(\varrho)\right\}$$

and

$$\xi^{(i)}_P(\varrho) = \max\left\{\xi^{(i)}_{P_1}(\varrho), \xi^{(i)}_{P_2}(\varrho)\right\}.$$

Definition 3.9. The *sum* of two PFM-sets

$$P_1 = \left\{ \frac{\varrho}{\left(\left\{\zeta_{P_1}^{(i)}(\varrho)\right\}_{i=1}^{n}\right), \left(\left\{\xi_{P_1}^{(i)}(\varrho)\right\}_{i=1}^{n}\right)} : \varrho \in X \right\}$$

and

$$P_2 = \left\{ \frac{\varrho}{\left(\left\{\zeta_{P_2}^{(i)}(\varrho)\right\}_{i=1}^{n}\right), \left(\left\{\xi_{P_2}^{(i)}(\varrho)\right\}_{i=1}^{n}\right)} : \varrho \in X \right\}$$

drawn from X is defined as

$$P_1 \oplus P_2 = \left\{ \frac{\varrho}{\left(\left\{\zeta_{P}^{(i)}(\varrho)\right\}_{i=1}^{n}\right), \left(\left\{\xi_{P}^{(i)}(\varrho)\right\}_{i=1}^{n}\right)} : \varrho \in X \right\}$$

where

$$\zeta_P^{(i)}(\varrho) = \zeta_{P_1}^{(i)}(\varrho) + \zeta_{P_2}^{(i)}(\varrho) - \zeta_{P_1}^{(i)}(\varrho)\zeta_{P_2}^{(i)}(\varrho)$$

and

$$\xi_P^{(i)}(\varrho) = \xi_{P_1}^{(i)}(\varrho)\xi_{P_2}^{(i)}(\varrho).$$

Definition 3.10. The *product* of two PFM-sets

$$P_1 = \left\{ \frac{\varrho}{\left(\left\{\zeta_{P_1}^{(i)}(\varrho)\right\}_{i=1}^{n}\right), \left(\left\{\xi_{P_1}^{(i)}(\varrho)\right\}_{i=1}^{n}\right)} : \varrho \in X \right\}$$

and

$$P_2 = \left\{ \frac{\varrho}{\left(\left\{\zeta_{P_2}^{(i)}(\varrho)\right\}_{i=1}^{n}\right), \left(\left\{\xi_{P_2}^{(i)}(\varrho)\right\}_{i=1}^{n}\right)} : \varrho \in X \right\}$$

drawn from X is defined as

$$P_1 \otimes P_2 = \left\{ \frac{\varrho}{\left(\left\{\zeta_{P}^{(i)}(\varrho)\right\}_{i=1}^{n}\right), \left(\left\{\xi_{P}^{(i)}(\varrho)\right\}_{i=1}^{n}\right)} : \varrho \in X \right\}$$

where

$$\zeta_P^{(i)}(\varrho) = \zeta_{P_1}^{(i)}(\varrho)\zeta_{P_2}^{(i)}(\varrho)$$

and

$$\xi_P^{(i)}(\varrho) = \xi_{P_1}^{(i)}(\varrho) + \xi_{P_2}^{(i)}(\varrho) - \xi_{P_1}^{(i)}(\varrho)\xi_{P_2}^{(i)}(\varrho).$$

Definition 3.11. The *difference* of two PFM-sets

$$P_1 = \left\{ \frac{\varrho}{\left(\left\{\zeta_{P_1}^{(i)}(\varrho)\right\}_{i=1}^{n}\right), \left(\left\{\xi_{P_1}^{(i)}(\varrho)\right\}_{i=1}^{n}\right)} : \varrho \in X \right\}$$

and

$$P_2 = \left\{ \frac{\varrho}{\left(\left\{\zeta_{P_2}^{(i)}(\varrho)\right\}_{i=1}^{n}\right), \left(\left\{\xi_{P_2}^{(i)}(\varrho)\right\}_{i=1}^{n}\right)} : \varrho \in X \right\}$$

drawn from X is defined as

$$P_1 \setminus P_2 = \left\{ \frac{\varrho}{\left(\left\{\zeta_{P}^{(i)}(\varrho)\right\}_{i=1}^{n}\right), \left(\left\{\xi_{P}^{(i)}(\varrho)\right\}_{i=1}^{n}\right)} : \varrho \in X \right\}$$

where

$$\zeta_P^{(i)}(\varrho) = \min\left\{\zeta_{P_1}^{(i)}(\varrho), \xi_{P_2}^{(i)}(\varrho)\right\}$$

and

$$\xi_P^{(i)}(\varrho) = \max\left\{\xi_{P_1}^{(i)}(\varrho), \zeta_{P_2}^{(i)}(\varrho)\right\}.$$

This difference is also termed as the *relative complement* of P_2 with respect to P_1.

Definition 3.12. The *symmetric difference* of two PFM-sets

$$P_1 = \left\{\frac{\varrho}{\left(\left\{\zeta_{P_1}^{(i)}(\varrho)\right\}_{i=1}^n\right), \left(\left\{\xi_{P_1}^{(i)}(\varrho)\right\}_{i=1}^n\right)} : \varrho \in X\right\}$$

and

$$P_2 = \left\{\frac{\varrho}{\left(\left\{\zeta_{P_2}^{(i)}(\varrho)\right\}_{i=1}^n\right), \left(\left\{\xi_{P_2}^{(i)}(\varrho)\right\}_{i=1}^n\right)} : \varrho \in X\right\}$$

drawn from X is defined as

$$P_1 \triangle P_2 \left\{\frac{\varrho}{\left(\left\{\zeta_{P}^{(i)}(\varrho)\right\}_{i=1}^n\right), \left(\left\{\xi_{P}^{(i)}(\varrho)\right\}_{i=1}^n\right)} : \varrho \in X\right\}$$

where

$$\zeta_P^{(i)}(\varrho) = \max\left\{\min\left\{\zeta_{P_1}^{(i)}(\varrho), \xi_{P_2}^{(i)}(\varrho)\right\}, \min\left\{\zeta_{P_2}^{(i)}(\varrho), \xi_{P_1}^{(i)}(\varrho)\right\}\right\}$$

and

$$\xi_P^{(i)}(\varrho) = \min\left\{\max\left\{\zeta_{P_1}^{(i)}(\varrho), \xi_{P_2}^{(i)}(\varrho)\right\}, \max\left\{\zeta_{P_2}^{(i)}(\varrho), \xi_{P_1}^{(i)}(\varrho)\right\}\right\}.$$

It is pertinent to observe that if we think of "max" as $\vee$ and "min" as $\wedge$, then the symmetric difference $P_1 \triangle P_2$ corresponds to XOR operation in Boolean logic.

Example 3.13. Let

$$\begin{aligned}
P_1 &= \left\{\frac{\varrho_1}{(0.82, 0.25), (0.19, 0.76)}, \frac{\varrho_2}{(0.71, 0.39, 0.16), (0.42, 0.39, 0.69)}, \frac{\varrho_3}{(0.88, 0.57), (0.19, 0.47)}\right\} \\
P_2 &= \left\{\frac{\varrho_1}{(0.66, 0.36), (0.41, 0.72)}, \frac{\varrho_2}{(0.67, 0.13, 0.11), (0.61, 0.56, 0.49)}, \frac{\varrho_3}{(0.10, 0.09), (0.86, 0.23)}\right\}
\end{aligned}$$

be PFM-sets drawn from $X = \{\varrho_1, \varrho_2, \varrho_3\}$, then

$$\begin{aligned}
P_1 \sqcup P_2 &= \left\{\frac{\varrho_1}{(0.82, 0.36), (0.19, 0.72)}, \frac{\varrho_2}{(0.71, 0.39, 0.16), (0.42, 0.39, 0.49)}, \frac{\varrho_3}{(0.88, 0.57), (0.19, 0.23)}\right\} \\
P_1 \sqcap P_2 &= \left\{\frac{\varrho_1}{(0.66, 0.25), (0.41, 0.76)}, \frac{\varrho_2}{(0.67, 0.13, 0.11), (0.61, 0.56, 0.69)}, \frac{\varrho_3}{(0.10, 0.09), (0.86, 0.47)}\right\} \\
P_1 \oplus P_2 &= \left\{\frac{\varrho_1}{(0.94, 0.52), (0.08, 0.55)}, \frac{\varrho_2}{(0.90, 0.47, 0.25), (0.26, 0.22, 0.34)}, \frac{\varrho_3}{(0.89, 0.61), (0.16, 0.11)}\right\} \\
P_1 \otimes P_2 &= \left\{\frac{\varrho_1}{(0.54, 0.09), (0.52, 0.93)}, \frac{\varrho_2}{(0.48, 0.05, 0.02), (0.77, 0.73, 0.84)}, \frac{\varrho_3}{(0.09, 0.05), (0.89, 0.59)}\right\} \\
P_1 \setminus P_2 &= \left\{\frac{\varrho_1}{(0.41, 0.25), (0.66, 0.76)}, \frac{\varrho_2}{(0.61, 0.39, 0.16), (0.67, 0.39, 0.69)}, \frac{\varrho_3}{(0.86, 0.23), (0.19, 0.47)}\right\} \\
P_1 \triangle P_2 &= \left\{\frac{\varrho_1}{(0.41, 0.36), (0.66, 0.72)}, \frac{\varrho_2}{(0.61, 0.13, 0.16), (0.67, 0.39, 0.49)}, \frac{\varrho_3}{(0.86, 0.23), (0.19, 0.47)}\right\}
\end{aligned}$$

Definition 3.14. The *complement* of the PFM-set

$$P = \left\{ \frac{\varrho}{\left(\{\zeta_P^{(i)}(\varrho)\}_{i=1}^{n}\right), \left(\{\xi_P^{(i)}(\varrho)\}_{i=1}^{n}\right)} : \varrho \in X \right\}$$

is defined as

$$P^c = \left\{ \frac{\varrho}{\left(\{\xi_P^{(i)}(\varrho)\}_{i=1}^{n}\right), \left(\{\zeta_P^{(i)}(\varrho)\}_{i=1}^{n}\right)} : \varrho \in X \right\}.$$

Example 3.15. The complement of PFM-set cited in Example 3.2 is

$$P^c = \left\{ \frac{\varrho_1}{(0.88, 0.56), (0.27, 0.12)}, \frac{\varrho_2}{(0.67, 0.62, 0.56), (0.11, 0.53, 0.24)}, \frac{\varrho_3}{(0.67, 0.17), (0.57, 0.91)} \right\}.$$

Observe that we shuffled the entries of ζ (and obviously the corresponding entries in ξ) so that the elements of ζ emerge as a descending sequence.

Theorem 3.16. *If P_1 and P_2 are PFM-sets drawn from X such that $P_1 \sqsubseteq P_2$, then $P_2^c \sqsubseteq P_1^c$.*

Proof. Since $P_1 \sqsubseteq P_2$, so $\zeta_{P_1}^{(i)} \leq \zeta_{P_2}^{(i)}$ and $\xi_{P_1}^{(i)} \geq \xi_{P_2}^{(i)}$. By definition, the sequence of membership and non-membership count for P_1^c is $\left(\xi_{P_1}^{(i)}, \zeta_{P_1}^{(i)}\right)$ and for P_2^c is $\left(\xi_{P_2}^{(i)}, \zeta_{P_2}^{(i)}\right)$ which quickly yields the desired result. □

Theorem 3.17. *If P_1 and P_2 are PFM-sets drawn from X such that $P_1 \setminus P_2 = P_1 \sqcap P_2^c$.*

Proof. Let

$$P_1 = \left\{ \frac{\varrho}{\left(\{\zeta_{P_1}^{(i)}(\varrho)\}_{i=1}^{n}\right), \left(\{\xi_{P_1}^{(i)}(\varrho)\}_{i=1}^{n}\right)} : \varrho \in X \right\}$$

$$P_2 = \left\{ \frac{\varrho}{\left(\{\zeta_{P_2}^{(i)}(\varrho)\}_{i=1}^{n}\right), \left(\{\xi_{P_2}^{(i)}(\varrho)\}_{i=1}^{n}\right)} : \varrho \in X \right\}.$$

Then, by definition

$$\begin{aligned} P_1 \sqcap P_2^c &= \left\{ \frac{\varrho}{\zeta_{P_1}^{(i)}(\varrho)\ {}_{i=1}^{n},\ \ \xi_{P_1}^{(i)}(\varrho)\ {}_{i=1}^{n}} : \varrho \in X \right\} \sqcap \left\{ \frac{\varrho}{\xi_{P_2}^{(i)}(\varrho)\ {}_{i=1}^{n},\ \ \zeta_{P_2}^{(i)}(\varrho)\ {}_{i=1}^{n}} : \varrho \in X \right\} \\ &= \left\{ \frac{\varrho}{\min\ \zeta_{P_1}^{(i)}(\varrho), \xi_{P_2}^{(i)}(\varrho)\ {}_{i=1}^{n},\ \ \max\ \xi_{P_1}^{(i)}(\varrho), \zeta_{P_2}^{(i)}(\varrho)\ {}_{i=1}^{n}} : \varrho \in X \right\} \\ &= P_1 \setminus P_2. \end{aligned}$$

□

Theorem 3.18. *If P_1 and P_2 are two PFM-sets drawn from X, then the idempotent and commutative laws hold under the operations of $\sqcup$ and $\sqcap$ i.e.*

(i) $P_1 \sqcup P_1 = P_1$
(ii) $P_1 \sqcap P_1 = P_1$
(iii) $P_1 \sqcup P_2 = P_2 \sqcup P_1$
(iv) $P_1 \sqcap P_2 = P_2 \sqcap P_1$.

Proof. Follows immediately from definitions of $\sqcup$ and $\sqcap$. □

Theorem 3.19. *If P_1 and P_2 are PFM-sets drawn from X, then De Morgan's laws hold i.e.*

(i) $(P_1 \sqcup P_2)^c = P_1^c \sqcap P_2^c$

(ii) $(P_1 \sqcap P_2)^c = P_1^c \sqcup P_2^c$.

Proof. Let

$$\begin{aligned} P_1 &= \left\{ \frac{\varrho}{\big(\{\zeta_{P_1}^{(i)}(\varrho)\}_{i=1}^{n}\big), \big(\{\xi_{P_1}^{(i)}(\varrho)\}_{i=1}^{n}\big)} : \varrho \in X \right\} \\ P_2 &= \left\{ \frac{\varrho}{\big(\{\zeta_{P_2}^{(i)}(\varrho)\}_{i=1}^{n}\big), \big(\{\xi_{P_2}^{(i)}(\varrho)\}_{i=1}^{n}\big)} : \varrho \in X \right\} \end{aligned}$$

be two PFM-sets defined over X, then

$$\begin{aligned} P_1 \sqcup P_2 &= \left\{ \frac{\varrho}{\big(\max\{\zeta_{P_1}^{(i)}(\varrho), \zeta_{P_2}^{(i)}(\varrho)\}_{i=1}^{n}\big), \big(\min\{\xi_{P_1}^{(i)}(\varrho), \xi_{P_2}^{(i)}(\varrho)\}_{i=1}^{n}\big)} : \varrho \in X \right\} \\ \Rightarrow (P_1 \sqcup P_2)^c &= \left\{ \frac{\varrho}{\big(\min\{\xi_{P_1}^{(i)}(\varrho), \xi_{P_2}^{(i)}(\varrho)\}_{i=1}^{n}\big), \big(\max\{\zeta_{P_1}^{(i)}(\varrho), \zeta_{P_2}^{(i)}(\varrho)\}_{i=1}^{n}\big)} : \varrho \in X \right\} \\ &= P_1^c \sqcap P_2^c \end{aligned}$$

which establishes (i). The proof of (ii) is parallel. □

Theorem 3.20. *If P_1 and P_2 are PFM-sets drawn from X, then*

(i) $(P_1 \oplus P_2)^c = P_1^c \otimes P_2^c$

(ii) $(P_1 \otimes P_2)^c = P_1^c \oplus P_2^c$.

Proof. Let

$$\begin{aligned} P_1 &= \left\{ \frac{\varrho}{\big(\{\zeta_{P_1}^{(i)}(\varrho)\}_{i=1}^{n}\big), \big(\{\xi_{P_1}^{(i)}(\varrho)\}_{i=1}^{n}\big)} : \varrho \in X \right\} \\ P_2 &= \left\{ \frac{\varrho}{\big(\{\zeta_{P_2}^{(i)}(\varrho)\}_{i=1}^{n}\big), \big(\{\xi_{P_2}^{(i)}(\varrho)\}_{i=1}^{n}\big)} : \varrho \in X \right\} \end{aligned}$$

be two PFM-sets defined over X, then

$$\begin{aligned} P_1 \oplus P_2 &= \left\{ \frac{\varrho}{\zeta_{P_1}^{(i)}(\varrho) + \zeta_{P_2}^{(i)}(\varrho) - \zeta_{P_1}^{(i)}(\varrho)\zeta_{P_2}^{(i)}(\varrho) \;{}_{i=1}^{n}, \quad \xi_{P_1}^{(i)}(\varrho)\xi_{P_2}^{(i)}(\varrho) \;{}_{i=1}^{n}} : \varrho \in X \right\} \\ \Rightarrow (P_1 \oplus P_2)^c &= \left\{ \frac{\varrho}{\xi_{P_1}^{(i)}(\varrho)\xi_{P_2}^{(i)}(\varrho) \;{}_{i=1}^{n}, \quad \zeta_{P_1}^{(i)}(\varrho) + \zeta_{P_2}^{(i)}(\varrho) - \zeta_{P_1}^{(i)}(\varrho)\zeta_{P_2}^{(i)}(\varrho) \;{}_{i=1}^{n}} : \varrho \in X \right\} \\ &= \left\{ \frac{\varrho}{\xi_{P_1}^{(i)}(\varrho) \;{}_{i=1}^{n}, \quad \zeta_{P_1}^{(i)}(\varrho) \;{}_{i=1}^{n}} : \varrho \in X \right\} \otimes \left\{ \frac{\varrho}{\xi_{P_2}^{(i)}(\varrho) \;{}_{i=1}^{n}, \quad \zeta_{P_2}^{(i)}(\varrho) \;{}_{i=1}^{n}} : \varrho \in X \right\} \\ &= P_1^c \otimes P_2^c \end{aligned}$$

which proves (i). The proof of (ii) may be furnished in the same fashion. □

Theorem 3.21. *If P_1, P_2 and P_3 are PFM-sets drawn from X, then*

(i) $P_1 \sqcup (P_2 \sqcup P_3) = (P_1 \sqcup P_2) \sqcup P_3$

(ii) $P_1 \sqcap (P_2 \sqcap P_3) = (P_1 \sqcap P_2) \sqcap P_3$

(iii) $P_1 \sqcup (P_2 \sqcap P_3) = (P_1 \sqcup P_2) \sqcap (P_1 \sqcup P_3)$
(iv) $P_1 \sqcap (P_2 \sqcup P_3) = (P_1 \sqcap P_2) \sqcup (P_1 \sqcap P_3)$.

Proof. Let

$$\begin{aligned}
P_1 &= \left\{ \frac{\varrho}{\left(\left\{\zeta_{P_1}^{(i)}(\varrho)\right\}_{i=1}^{n}\right), \left(\left\{\xi_{P_1}^{(i)}(\varrho)\right\}_{i=1}^{n}\right)} : \varrho \in X \right\} \\
P_2 &= \left\{ \frac{\varrho}{\left(\left\{\zeta_{P_2}^{(i)}(\varrho)\right\}_{i=1}^{n}\right), \left(\left\{\xi_{P_2}^{(i)}(\varrho)\right\}_{i=1}^{n}\right)} : \varrho \in X \right\} \\
P_3 &= \left\{ \frac{\varrho}{\left(\left\{\zeta_{P_3}^{(i)}(\varrho)\right\}_{i=1}^{n}\right), \left(\left\{\xi_{P_3}^{(i)}(\varrho)\right\}_{i=1}^{n}\right)} : \varrho \in X \right\}
\end{aligned}$$

be PFM-sets defined over X, then

$$\begin{aligned}
P_1 \sqcup (P_2 \sqcup P_3) &= \left\{ \frac{\varrho}{\max\ \zeta_{P_1}^{(i)}, \max\{\zeta_{P_2}^{(i)}, \zeta_{P_3}^{(i)}\}\ {}_{i=1}^{\ n}\ ,\ \min\ \xi_{P_1}^{(i)}\ , \min\{\xi_{P_2}^{(i)}, \xi_{P_3}^{(i)}\}\ {}_{i=1}^{\ n}} : \varrho \in X \right\} \\
&= \left\{ \frac{\varrho}{\max\ \zeta_{P_1}^{(i)}, \zeta_{P_2}^{(i)}, \zeta_{P_3}^{(i)}\ {}_{i=1}^{\ n}\ ,\ \min\ \xi_{P_1}^{(i)}, \xi_{P_2}^{(i)}, \xi_{P_3}^{(i)}\ {}_{i=1}^{\ n}} : \varrho \in X \right\} \\
&= \left\{ \frac{\varrho}{\max\ \max\{\zeta_{P_1}^{(i)}, \zeta_{P_2}^{(i)}\}, \zeta_{P_3}^{(i)}\ {}_{i=1}^{\ n}\ ,\ \min\ \min\{\xi_{P_1}^{(i)}, \xi_{P_2}^{(i)}\}, \xi_{P_3}^{(i)}\ {}_{i=1}^{\ n}} : \varrho \in X \right\} \\
&= (P_1 \sqcup P_2) \sqcup P_3.
\end{aligned}$$

This establishes (i). The proof of (ii) is parallel.
Now, we head towards proving (iv). The proof of (iii) may be furnished on the parallel track.
For $\varrho \in X$, we have
$P_1 \sqcap (P_2 \sqcup P_3)$

$$= \left\{ \frac{\varrho}{\left(\min\left\{\zeta_{P_1}^{(i)}, \max\{\zeta_{P_2}^{(i)}, \zeta_{P_3}^{(i)}\}\right\}_{i=1}^{n}\right), \left(\max\left\{\xi_{P_1}^{(i)}, \min\{\xi_{P_2}^{(i)}, \xi_{P_3}^{(i)}\}\right\}_{i=1}^{n}\right)} \right\}$$

$$= \left\{ \frac{\varrho}{\left(\max\{\{\min\{\zeta_{P_1}^{(i)}, \zeta_{P_2}^{(i)}\}, \min\{\zeta_{P_1}^{(i)}, \zeta_{P_3}^{(i)}\}\}\}_{i=1}^{n}\right), \left(\min\{\{\max\{\xi_{P_1}^{(i)}, \xi_{P_2}^{(i)}\}, \max\{\xi_{P_1}^{(i)}, \xi_{P_3}^{(i)}\}\}\}_{i=1}^{n}\right)} \right\}$$

$= (P_1 \sqcap P_2) \sqcup (P_1 \sqcap P_3)$. $\square$

Theorem 3.22. *If P_1, P_2 and P_3 are PFM-sets drawn from X, then*

(i) $P_1 \oplus (P_2 \sqcup P_3) = (P_1 \oplus P_2) \sqcup (P_1 \oplus P_2)$
(ii) $P_1 \oplus (P_2 \sqcap P_3) = (P_1 \oplus P_2) \sqcap (P_1 \oplus P_2)$
(iii) $P_1 \otimes (P_2 \sqcup P_3) = (P_1 \otimes P_2) \sqcup (P_1 \otimes P_2)$
(iv) $P_1 \otimes (P_2 \sqcap P_3) = (P_1 \otimes P_2) \sqcap (P_1 \otimes P_2)$.

Proof. We prove (i) here. The other assertions may be proved employing the similar argument. Assume that

$$\begin{aligned} P_1 &= \left\{ \frac{\varrho}{\left(\{\zeta^{(i)}_{P_1}(\varrho)\}_{i=1}^{n}\right), \left(\{\xi^{(i)}_{P_1}(\varrho)\}_{i=1}^{n}\right)} : \varrho \in X \right\} \\ P_2 &= \left\{ \frac{\varrho}{\left(\{\zeta^{(i)}_{P_2}(\varrho)\}_{i=1}^{n}\right), \left(\{\xi^{(i)}_{P_2}(\varrho)\}_{i=1}^{n}\right)} : \varrho \in X \right\} \\ P_3 &= \left\{ \frac{\varrho}{\left(\{\zeta^{(i)}_{P_3}(\varrho)\}_{i=1}^{n}\right), \left(\{\xi^{(i)}_{P_3}(\varrho)\}_{i=1}^{n}\right)} : \varrho \in X \right\} \end{aligned}$$

be PFM-sets defined over X, then for $\varrho \in X$

$$\begin{aligned} P_1 \oplus (P_2 \sqcup P_3) &= \left\{ \frac{\varrho}{\zeta^{(i)}_{P_1} + \max\ \zeta^{(i)}_{P_2}, \zeta^{(i)}_{P_3}\ - \zeta^{(i)}_{P_1} \max\ \zeta^{(i)}_{P_2}, \zeta^{(i)}_{P_3}\ {}_{i=1}^{n}\ ,\ \ \xi^{(i)}_{P_1} \min\ \xi^{(i)}_{P_2}, \xi^{(i)}_{P_3}\ {}_{i=1}^{n}} \right\} \\ &= \left\{ \frac{\varrho}{\max\ \zeta^{(i)}_{P_1} + \zeta^{(i)}_{P_2} - \zeta^{(i)}_{P_1}\zeta^{(i)}_{P_2}, \zeta^{(i)}_{P_1} + \zeta^{(i)}_{P_3} - \zeta^{(i)}_{P_1}\zeta^{(i)}_{P_3}\ {}_{i=1}^{n}\ ,\ \ \xi^{(i)}_{P_1} \min\ \xi^{(i)}_{P_2}\xi^{(i)}_{P_3}\ {}_{i=1}^{n}} \right\} \\ &= (P_1 \oplus P_2) \sqcup (P_1 \oplus P_2). \end{aligned}$$

$\square$

Definition 3.23. The *Cartesian product* of two PFM-sets

$$P_1 = \left\{ \frac{\varrho}{\left(\zeta^{(1)}_{P_1}(\varrho), \zeta^{(2)}_{P_1}(\varrho), \cdots, \zeta^{(n)}_{P_1}(\varrho)\right), \left(\xi^{(1)}_{P_1}(\varrho), \xi^{(2)}_{P_1}(\varrho), \cdots, \xi^{(n)}_{P_1}(\varrho)\right)} : \varrho \in X \right\}$$

and

$$P_2 = \left\{ \frac{\varrho'}{\left(\zeta^{(1)}_{P_2}(\varrho'), \zeta^{(2)}_{P_2}(\varrho), \cdots, \zeta^{(n)}_{P_2}(\varrho')\right), \left(\xi^{(1)}_{P_2}(\varrho'), \xi^{(2)}_{P_2}(\varrho'), \cdots, \xi^{(n)}_{P_2}(\varrho')\right)} : \varrho' \in X \right\}$$

drawn from X is defined as

$$P_1 \times P_2 = \left\{ \frac{(\varrho, \varrho')}{\zeta^{(1)}_{P}(\varrho)\zeta^{(1)}_{P}(\varrho'), \cdots, \zeta^{(n)}_{P}(\varrho)\zeta^{(n)}_{P}(\varrho')\ ,\ \xi^{(1)}_{P}(\varrho)\xi^{(1)}_{P}(\varrho'), \cdots, \xi^{(n)}_{P}(\varrho)\xi^{(n)}_{P}(\varrho')} : \varrho, \varrho' \in X \right\}.$$

Example 3.24. Let

$$\begin{aligned} P_1 &= \left\{ \frac{\varrho_1}{(0.38, 0.29), (0.17, 0.16)}, \frac{\varrho_2}{(1, 0.16), (0, 0.69)} \right\} \\ P_2 &= \left\{ \frac{\varrho_1}{(0.67, 0.43), (0.42, 0.44)}, \frac{\varrho_2}{(0.13, 0.13), (0.61, 0.49)} \right\} \end{aligned}$$

be PFM-sets drawn from $X = \{\varrho_1, \varrho_2\}$, then

$$P_1 \times P_2 = \left\{ \frac{(\varrho_1, \varrho_1)}{(0.25, 0.12), (0.07, 0.07)}, \frac{(\varrho_1, \varrho_2)}{(0.05, 0.04), (0.10, 0.08)}, \frac{(\varrho_2, \varrho_1)}{(0.67, 0.07), (0, 0.30)}, \frac{(\varrho_2, \varrho_2)}{(0.13, 0.02), (0, 0.34)} \right\}.$$

Theorem 3.25. *If P_1, P_2 and P_3 are PFM-sets drawn from X, then*

(i) $P_1 \times P_2 = P_2 \times P_1$

(ii) $P_1 \times (P_2 \times P_3) = (P_1 \times P_2) \times P_3$

(iii) $P_1 \times (P_2 \sqcup P_3) = (P_1 \times P_2) \sqcup (P_1 \times P_3)$
(iv) $P_1 \times (P_2 \sqcap P_3) = (P_1 \times P_2) \sqcap (P_1 \times P_3)$
(v) $P_1 \times (P_2 \oplus P_3) \sqsubseteq (P_1 \times P_2) \oplus (P_1 \times P_3)$
(vi) $P_1 \times (P_2 \otimes P_3) \sqsupseteq (P_1 \times P_2) \otimes (P_1 \times P_3)$.

Proof. Straight forward. □

Definition 3.26. The distance between two PFM-sets P_1 and P_2 of X is defined in different ways as:

(1) The *Euclidean distance*:

$$d_E(P_1, P_2) = \sqrt{\frac{1}{2}\Sigma_{j=1}^{n}\left\{\left(\zeta_{P_1}^{(i)}(\varrho_j) - \zeta_{P_2}^{(i)}(\varrho_j)\right)^2 + \left(\xi_{P_1}^{(i)}(\varrho_j) - \xi_{P_2}^{(i)}(\varrho_j)\right)^2\right\}}.$$

(2) The *Hamming distance*:

$$d_H(P_1, P_2) = \frac{1}{2}\Sigma_{j=1}^{n}\Big(|\zeta_{P_1}^{(i)}(\varrho_j) - \zeta_{P_2}^{(i)}(\varrho_j)| + |\xi_{P_1}^{(i)}(\varrho_j) - \xi_{P_2}^{(i)}(\varrho_j)|\Big).$$

(3) The *normalized Hamming distance*:

$$d_{n-H}(P_1, P_2) = \frac{1}{2n}\Sigma_{j=1}^{n}\Big(|\zeta_{P_1}^{(i)}(\varrho_j) - \zeta_{P_2}^{(i)}(\varrho_j)| + |\xi_{P_1}^{(i)}(\varrho_j) - \xi_{P_2}^{(i)}(\varrho_j)|\Big).$$

(4) The *normalized Euclidean distance*:

$$d_{n-E}(P_1, P_2) = \sqrt{\frac{1}{2n}\Sigma_{j=1}^{n}\Big((\zeta_{P_1}^{(i)}(\varrho_j) - \zeta_{P_2}^{(i)}(\varrho_j))^2 + (\xi_{P_1}^{(i)}(\varrho_j) - \xi_{P_2}^{(i)}(\varrho_j))^2\Big)}.$$

for each admissible value of i.

4. Multi-Attribute Group Decision Making based on PFM-Sets in Medical Diagnosis

Decision making is a critical technique for deciding on most appropriate choice from available alternatives. By means of stage by stage course of decision-making assists us making more purposive and intellectual decisions by framing pertinent data illustrating alternatives. In this section, we present an application of multi-attribute group decision making (MAGDM) in medical diagnosis. MAGDM is a process that arranges for an undisputed and unanimous assessment of a number of professionals based on priorities, data values and beliefs of the decision makers collectively to assign ranking to the alternatives and to attain the best truthful solution to different real world problems.

Case Study:

Diabetes mellitus, commonly known as simply diabetes, is a severe condition in which patient can't perform its life activities normally. In its chronic condition it badly affects vision, vascular and excretory system physiologically. Three types of diabetes are commonly identified in the patients including Insulin Dependent Diabetes Mellitus (IDDM, Type-1), Non Insulin Dependent Diabetes Mellitus (NIDDM, Type-2) and GDM (Gestational Diabetes Mellitus). In all these types of diabetes there is a high level of glucose in the patient which disturbs the osmotic balance in the blood.

Glucose is produced in our bodies after the digestion of food especially carbohydrates. A part of this glucose is utilized by the body as a fuel to produce energy and rest of it is converted to glycogen or proteins and is stored in the liver and muscle cells. This conversion is controlled by a pancreatic hormone, insulin. Deficiency of this hormone or no response to the hormone leads to constant increase in the level of glucose in the blood.

IDDM or Type-1 Diabetes:

IDDM is also commonly known as juvenile diabetes for it occurs before the age of 30. In this type, antibodies of the patient's body destroy β cells of the pancreas which are basically involved in the production of insulin hormone. Other body organs that can be affected by diabetes of type-1 include eyes, kidney, heart and brain. Diabetes of type-1 can be cured by taking insulin. Insulin can be injected by syringes, insulin pens and insulin pumps. Glycosylated hemoglobin test (A1C) can help to identify overall glucose level control over past three months. If someone is suffering from diabetes of type-1, he or she can prolong his or her life by certain changes in the life style such as regular exercise, careful meal planning and taking medicines according to treatment plan.

NIDDM or Type-2 Diabetes:

Nearly 95 % of diabetic patients suffer from type-2 diabetes. It is also called non insulin dependent diabetes. Health risk associated with type-2 diabetes are almost same but of low intensity. During diabetes of type-2, the insulin produced is either too small or body cells especially muscle and liver cells are resistant to it. Being obese is an important factor causing NIDDM, where pancreas has to perform over activities for the production of insulin even then it remains insufficient for the body. Unfortunately, there is no proper cure of NIDDM. Patients have to manage their body weight to reduce the problem. For this purpose regular exercise, balanced diet and regular HbA1c test have been proving to be valuable tool to reduce the risk of NIDDM.

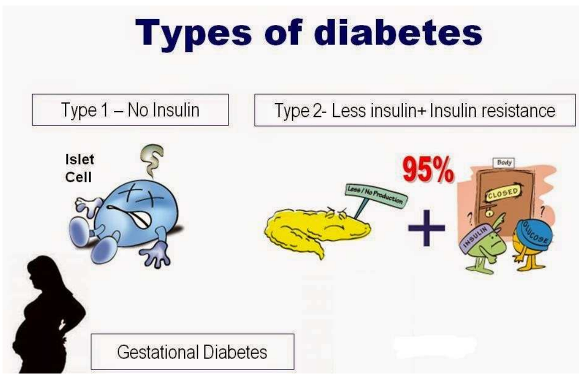

Figure 1. Types of diabetes

Gestational Diabetes Mellitus (GDM):

Gestation diabetes mellitus is related to the pregnancy, during which in some females (2-10 %) liver cells become least responsive to the insulin. After the birth of baby the problem mostly disappears automatically or a female (about 10 %) may suffers from type-2 diabetes later on. During the gestation period high glucose level in mother's blood stream may be severe for the developing baby. These problems include higher weight than normal and breathing issues after birth. On the other hand mother has to face certain kidney, heart, eyes and some nervous problems during this period. In case of baby over weight, normal birth is not possible and cesarean section remains an only one option for the birth. Regular exercise, balanced diet, use of insulin and counseling with health care experts have proved to be useful in reduction of GDM.

Other Forms of Diabetes:

About 1-5 % patients suffer from other types of diabetes, which may include infection in pancreas, side effects of certain drugs etc. Among these patients certain symptoms of diabetes like abnormal thirst, short interval urination, fatigue, gain or loss of body weight, nausea, poor wound healing, and skin contaminations etc. appear.

Now, we present an application in medical diagnosis employing Pythagorean fuzzy multisets. First we propose Algorithm 1 as given below.

Algorithm 1

Step 1: Choose the set of patients $P = \{p_1, p_2, \cdots, p_n\}$.

Step 2: Choose the set of diseases $D = \{d_1, d_2, \cdots, d_i\}$ and the set of symptoms $S = \{s_1, s_2, \cdots, s_j\}$.

Step 3: Construct table of PF-set of diseases vs symptoms & PFM-set of patients vs symptoms.

Step 4: Compute distance between patients & diseases employing any formula given in Definition 3.26.

Step 5: Optimal choice is the smallest distance between patient and disease.

Step 6: State the results in layman's language.

The flow chart explaining the procedural steps is depicted in Figure 2.

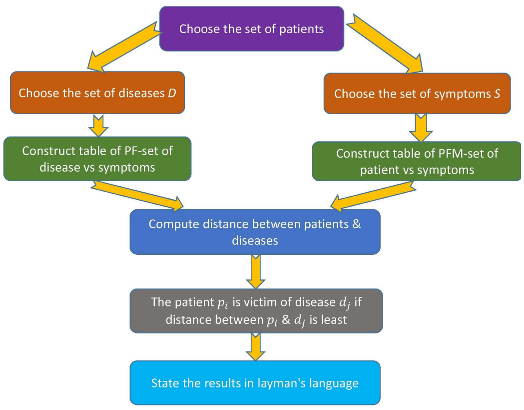

FIGURE 2. Flow chart representation of Algorithm 1

4.1. **Numerical Example.** Let $X = \{p_i : i = 1, 2, \cdots, 5\}$ be the set of patients under study, and $D = \{d_i : i = 1, 2, 3, 4\}$ be the set of types of diabetes, where

$$\begin{aligned} d_1 &= \text{Type-1 diabetes,} \\ d_2 &= \text{Type-2 diabetes,} \\ d_3 &= \text{Gestational diabetes, and} \\ d_4 &= \text{Other form of diabetes} \end{aligned}$$

Suppose that $S = \{s_i : i = 1, 2, \cdots, 6\}$ is the set of symptoms, where

$$\begin{aligned} s_1 &= \text{Short interval urination,} \\ s_2 &= \text{Gain/loss of body weight,} \\ s_3 &= \text{Dizziness or blurry vision,} \\ s_4 &= \text{Poor wound healing, and} \\ s_5 &= \text{Skin contaminations} \end{aligned}$$

Table 1 gives symptoms vs diabetes type, each symptom s_i is referred to by two numbers, namely, the membership ζ_i and non-membership ξ_i. The objective is to get an appropriate diagnosis for each patient.

	s_1	s_2	s_3	s_4	s_5
d_1	$(0.76, 0.32)$	$(0.59, 0.17)$	$(0.82, 0.24)$	$(0.39, 0.52)$	$(0.47, 0.72)$
d_2	$(0.65, 0.53)$	$(0.57, 0.43)$	$(0.27, 0.91)$	$(0.38, 0.64)$	$(0.37, 0.82)$
d_3	$(0.60, 0.12)$	$(0.13, 0.92)$	$(0.44, 0.83)$	$(0.79, 0.26)$	$(0.34, 0.32)$
d_4	$(0.12, 0.32)$	$(0.19, 0.52)$	$(0.28, 0.82)$	$(0.78, 0.24)$	$(0.89, 0.13)$

Table 1. PFNs showing symptoms vs type of diabetes

Table 2 yields information regarding patients vs symptoms in the standard form of representing PFM-sets i.e. $\big((\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)}), (\xi^{(1)}, \xi^{(2)}, \xi^{(3)})\big)$. The readings are taken at two different times to be more specific about type of diabetes.

	s_1	s_2	s_3	s_4	s_5
p_1	$(0.57, 0.92, 0.76)$ $(0.49, 0.18, 0.33)$	$(0.82, 0.77, 0.25)$ $(0.26, 0.28, 0.19)$	$(0.11, 0.23, 0.06)$ $(0.43, 0.44, 0.84)$	$(0.65, 0.47, 0.64)$ $(0.56, 0.25, 0.14)$	$(0.37, 0.23, 0.76)$ $(0.21, 0.58, 0.28)$
p_2	$(0.78, 0.89, 0.86)$ $(0.11, 0.12, 0.24)$	$(0.32, 0.13, 0.40)$ $(0.17, 0.11, 0.42)$	$(0.67, 0.45, 0.34)$ $(0.54, 0.49, 0.46)$	$(0.53, 0.46, 0.11)$ $(0.17, 0.59, 0.23)$	$(0.32, 0.56, 0.38)$ $(0.21, 0.55, 0.48)$
p_3	$(0.07, 0.24, 0.28)$ $(0.38, 0.41, 0.73)$	$(0.07, 0.21, 0.16)$ $(0.22, 0.23, 0.48)$	$(0.32, 0.29, 0.24)$ $(0.76, 0.77, 0.54)$	$(0.31, 0.29, 0.28)$ $(0.26, 0.26, 0.38)$	$(0.44, 0.43, 0.44)$ $(0.27, 0.44, 0.45)$
p_4	$(0.55, 0.59, 0.32)$ $(0.51, 0.62, 0.40)$	$(0.22, 0.34, 0.27)$ $(0.74, 0.36, 0.88)$	$(0.48, 0.44, 0.12)$ $(0.28, 0.45, 0.13)$	$(0.60, 0.39, 0.42)$ $(0.14, 0.35, 0.40)$	$(0.55, 0.36, 0.43)$ $(0.19, 0.46, 0.25)$
p_5	$(0.64, 0.41, 0.32)$ $(0.28, 0.37, 0.19)$	$(0.26, 0.33, 0.29)$ $(0.26, 0.67, 0.29)$	$(0.44, 0.44, 0.35)$ $(0.43, 0.40, 0.41)$	$(0.19, 0.28, 0.54)$ $(0.05, 0.20, 0.45)$	$(0.31, 0.29, 0.33)$ $(0.50, 0.47, 0.48)$

Table 2. PFM information showing symptoms vs type of diabetes

We employ normalized hamming distance with $n = 5$, which represents number of symptoms, to compute the distances between patients and diabetes shown in Table

3. The patient p_i having minimal distance with the diabetes d_j is likely to fall prey of d_j.

	d_1	d_2	d_3	d_4
p_1	0.769	**0.732**	0.826	0.854
p_2	**0.619**	0.794	0.678	0.944
p_3	0.901	0.710	0.746	**0.594**
p_4	0.806	0.678	**0.669**	0.793
p_5	0.699	0.748	**0.676**	0.836

TABLE 3. Distance between the patient p_i and the diabetes d_j

In view of above table, we conclude that the patient p_1 is likely to suffer from diabetes of type-2, p_2 from diabetes of type-1, p_3 from other forms of diabetes, whereas the patients p_4 and p_5 are victims of Gestational diabetes.
We depict these results with the help of Figure 3.

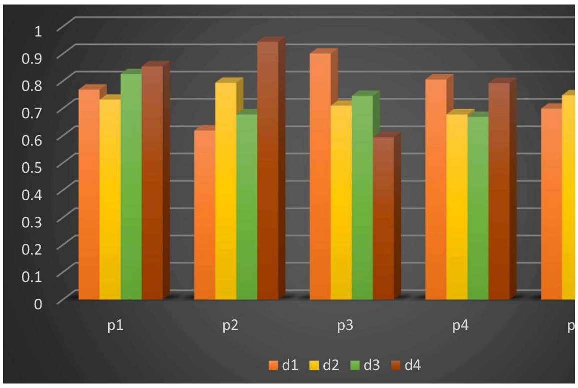

FIGURE 3. Distances between patient p_i and diabetes d_j

Since the height of the bar for the patient p_1 is least for the disease d_2, so the the patient p_1 is least distant from the diabetes of type-2 i.e. the patient p_1 is likely to fall a prey of diabetes of type-2. The rest of the chart may be interpreted on the parallel track.

5. Multi-Attribute Group Decision Making based on PFM-Sets in Pattern Recognition

In this section, we discuss multi-attribute group decision making (MAGDM) in pattern recognition. Pattern recognition (PR) is a scientific discipline that aims at classification of objects into a number of categories. This technique is successfully employed in detection of shapes, forms and classification of patterns in data. Popular areas of applications of PR are traffic analysis & control, speech recognition, classification of rocks, biological signals, smell recognition, interpreting DNA sequence, credit fraud detection, biometrics including finger prints, palm vein technology & face recognition, medical diagnosis, weather prediction, psychology,

computer science, voice to text changing, voice dialing, testing mental abilities/IQ level, behavior analysis, predicting complete words, ethology, terrorist detection, radar detection, and automated target recognition in military applications etc.
Now we propose Algorithm 2 that is given below.

Algorithm 2

Step 1: Construct the PFM-sets P_i's to be tested for PR.
Step 2: Construct PFM-set P from whom PR is to be tested.
Step 3: Construct table of average values for each P_i & P (optional).
Step 4: Compute distance between each P_i & P utilizing any formula given in Definition 3.26.
Step 5: Optimal choice is the P_i having smallest distance with P.
Step 6: Describe the results in layman's language.

The flow chart explaining the procedural steps of Algorithm 2 is portrayed in Figure 4.

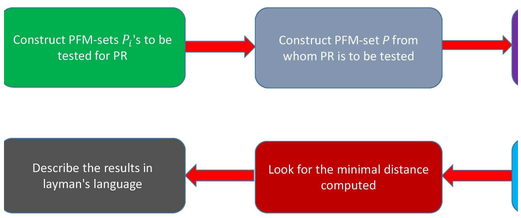

FIGURE 4. Flowchart of Algorithm 2

5.1. **Numerical Example.** Let $X = \{\varrho_i : i = 1, 2, \cdots, n\}$ be a crisp set having $X_1 = \{\varrho_1, \varrho_2\}$, $X_2 = \{\varrho_1, \varrho_4\}$ and $X_3 = \{\varrho_3, \varrho_4\}$. Let PFM-sets P_1, P_2 and P_3 be as given in Tables 4, 5 and 6 respectively.

P_1	ζ	ξ
	$(0.71, 0.56, 0.39)$	$(0.49, 0.67, 0.31)$
ϱ_1	$(0.53, 0.54, 0.46)$	$(0.52, 0.66, 0.29)$
	$(0.64, 0.61, 0.57)$	$(0.44, 0.56, 0.12)$
	$(0.94, 0.75, 0.73)$	$(0.22, 0.39, 0.45)$
ϱ_2	$(0.82, 0.78, 0.67)$	$(0.44, 0.53, 0.71)$
	$(0.76, 0.52, 0.29)$	$(0.34, 0.53, 0.59)$

TABLE 4. PFM-set P_1

P_2	ζ	ξ
ϱ_1	$(0.57, 0.41, 0.17)$	$(0.32, 0.76, 0.36)$
	$(0.65, 0.64, 0.63)$	$(0.74, 0.74, 0.75)$
	$(0.53, 0.43, 0.19)$	$(0.02, 0.17, 0.82)$
ϱ_4	$(0.92, 0.42, 0.07)$	$(0.16, 0.25, 0.04)$
	$(0.87, 0.76, 0.02)$	$(0.49, 0.40, 0.46)$
	$(0.45, 0.37, 0.36)$	$(0.43, 0.82, 0.06)$

Table 5. PFM-set P_2

P_3	ζ	ξ
ϱ_3	$(0.86, 0.43, 0.28)$	$(0.31, 0.52, 0.19)$
	$(0.99, 0.87, 0.54)$	$(0.14, 0.46, 0.16)$
	$(0.76, 0.73, 0.63)$	$(0.26, 0.40, 0.15)$
ϱ_4	$(0.51, 0.47, 0.26)$	$(0.43, 0.46, 0.89)$
	$(0.28, 0.24, 0.23)$	$(0.24, 0.28, 0.28)$
	$(0.77, 0.58, 0.53)$	$(0.35, 0.20, 0.75)$

Table 6. PFM-set P_3

Let pattern P of PFM-set be referred to as given in Table 7.

P	ζ	ξ
ϱ_9	$(0.35, 0.27, 0.26)$	$(0.82, 0.57, 0.82)$
	$(0.45, 0.44, 0.30)$	$(0.29, 0.01, 0.81)$
	$(0.76, 0.38, 0.37)$	$(0.31, 0.05, 0.49)$
ϱ_{10}	$(0.69, 0.64, 0.50)$	$(0.48, 0.40, 0.72)$
	$(1.00, 0.62, 0.59)$	$(0.00, 0.54, 0.38)$
	$(0.63, 0.38, 0.29)$	$(0.18, 0.87, 0.28)$

Table 7. PFM-set P

The tables of average values for the PFM-sets given in Tables 4, 5, 6 and 7, respectively, are given in order in Tables 8, 9, 10 and 11.

P_1	ζ	ξ
ϱ_1	$(0.627, 0.570, 0.473)$	$(0.483, 0.630, 0.240)$
ϱ_2	$(0.840, 0.683, 0.563)$	$(0.333, 0.483, 0.583)$

Table 8. Average values of PFM-set P_1

P_2	ζ	ξ
ϱ_1	$(0.583, 0.493, 0.330)$	$(0.360, 0.556, 0.643)$
ϱ_4	$(0.746, 0.516, 0.150)$	$(0.360, 0.490, 0.186)$

Table 9. Average values of PFM-set P_2

P_3	ζ	ξ
ϱ_3	$(0.870, 0.676, 0.483)$	$(0.236, 0.460, 0.167)$
ϱ_4	$(0.520, 0.430, 0.340)$	$(0.340, 0.313, 0.640)$

Table 10. Average values of PFM-set P_3

P	ζ	ξ
ϱ_9	$(0.520, 0.363, 0.310)$	$(0.473, 0.210, 0.706)$
ϱ_{10}	$(0.773, 0.546, 0.460)$	$(0.220, 0.603, 0.460)$

Table 11. Average values of PFM-set P

The Euclidean distances between P and P_i's are computed as

$$\begin{aligned} d_E(P, P_1) &= 0.7195, \\ d_E(P, P_2) &= \mathbf{0.6812}, \\ d_E(P, P_3) &= 0.8142. \end{aligned}$$

These distances are portrayed in Figure 5, which is self explanatory:

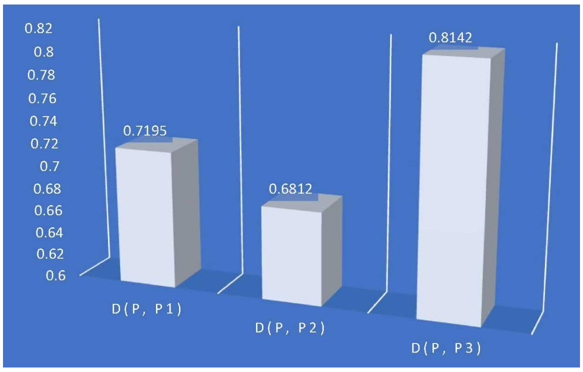

Figure 5. Euclidean distances between P and P_i's

Since the minimal distance is 0.6812, which is between the patterns P and P_2, so we conclude that the testing pattern P and the pattern P_2 follow the same pattern i.e. the pattern P_2 is likely to be pattern recognizable with the pattern P.

6. Comparative analysis and superiority of the proposed work

Fuzzy sets, intuitionistic fuzzy sets and Pythagorean fuzzy sets are important mathematical models to deal with uncertainties but these theories have their own limitations regarding membership grades, non-membership grades, space for membership and/or non-membership grades and multiplicity (repetition) in membership and non-membership grades. Decision analysis over some real world problems becomes limited while dealing with these theories. For example, IFS fails to deal with the situation when the sum of membership and non-membership grades exceeds unity. To solve this problem, PFS provides a larger space for membership and non-membership grades to the decision makers. Fuzzy sets deal with uncertainty but does not handle non-membership grades. None of IFS and PFS deal with multiplicities. The proposed model of Pythagorean fuzzy multisets is more flexible and suitable to deal with the situation when the decision makers have to assign arbitrary number of pairs of membership and non-membership grades to the alternatives in the larger space (The space in which sum of squares of membership and non-membership values do not exceed unity).
The proposed technique is more flexible and practical technique to deal with uncertainties and vagueness in the field of decision analysis. Table 12 shows that the existing methodologies have some drawbacks and limitations, which can be overcome by using the proposed technique. From Table 12, it is clear that Pythagorean fuzzy multiset is more suitable technique when the input data is available in the form of multiple (paired) information.

Model	Membership grade	Non-membership grade	Multiplicity in membership grades	Enlarged space
FS [62]	✓	×	×	×
IFS [6]	✓	✓	×	×
PF-set [60, 61]	✓	✓	×	✓
IFMS [51]	✓	×	✓	×
PFM-set (proposed)	✓	✓	✓	✓

Table 12. Comparison of Pythagorean fuzzy multiset with existing structures

7. Conclusion

We brought together the notion of Pythagorean fuzzy multisets in this clause. We presented various algebraic operations on PFM-sets including subset, union, intersection, sum, product, difference, symmetric difference, complement and Cartesian product. We also presented some peculiar mathematical properties of PFM-sets along with their proofs. A number of examples is included to comprehend the concepts proficiently. We have proposed two algorithms for modeling uncertainties in the multi-attribute group decision-making (MAGDM) by using PFM-sets. The proposed Algorithm 1 and Algorithm 2 based on PFM-sets have successfully applied for MAGDM of the real world problems including medical diagnosis & pattern recognition. We have explained the methodology by using flow charts. Brief but comprehensive detail of different types of diabetes and symptoms of these types, and the sectors where pattern recognition is employed is also included. Superiority of the proposed model over the contemporary models is also discussed with brevity. PFM-sets have tremendous potential for further exploration in theoretical besides application perspective. One can extend this work to some new models like Pythagorean fuzzy soft multiset, PFM N-soft, PFM N-soft expert set and PFM rough set to derive algebraic and topological structures on these hybrid models. These ideas may be efficiently employed in handling uncertainties in different sectors of real life situations including artificial intelligence, image processing, pattern recognition, medical diagnosis, forecasting, robotics and recruitment problems.

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Authors contributions
The authors contributed to each part of this paper equally. The authors read and approved the final manuscript.

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