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Application of Fixed Points in Differential Inclusions of Heat Conduction

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Abstract. In this article some new fixed point and hybrid coincidence point theorems for multivalued mappings satisfying generalized contractive conditions have been proved in generalized metric spaces. Some non-trivial examples are given to support our main theorems. As application, a differential inclusion problem for heat conduction in metals has been given, using our main result, we prove the existence of solutions to the given differential inclusion problem.

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Key Words: G-metric space; multi-valued mappings; fixed points; coincidence points; differential inclusions; heat conduction.

1. Introduction

The most excited results in the existence theory of linear and non-linear operators is systematically proved in 1922 by Banach. A systematic iterative procedure has been developed to prove the existence of a unique fixed point a self mapping in the settings of complete metric spaces. This result has attracted a huge number of researchers in finding out the solutions of many linear and non-linear physical problems. In theoretical aspect, it has great impact on the existence theory of partial and ordinary differential equations. Banach contraction principle has been generalized in various sense, like generalizing mappings, spaces, contractive conditions and combinations of any of these.

Generalizing the metric structure is also very important, and many generalizations have been presented in the literature. Some of the most common are; 2-metric space, D-metric space, b-metric space, fuzzy metric space, partial metric space, and many others including cone metric space [6, 9, 10, 14, 15, 16, 17, 20, 23, 41, 42, 45, 46, 49, 50]. Some of the flaws have been recovered by Mustafa and Sims [27, 28], and a new metric was defined known as G-metric space. Many fixed point results for single and multi-valued mappings have been proved in the settings of G-metric space [5, 12, 13, 19, 26, 29, 30, 31, 32, 33, 36, 38, 51]. Existence results in the theory of ordinary and partial differential equations are also proved using the results of G-metric space, for details, see [2, 3, 4, 7, 8, 11, 21, 22, 24, 25, 34, 35, 37].

In [18], it has been shown that many fixed point results in G-metric space are just the consequence of some classical fixed point results in standard metric space. Recently in [12], author proves some innovative fixed point results in G-metric spaces. Inspired by the results of [12], we initiated the study for finding fixed points of multi-valued mappings in the settings of complete G-metric spaces [38]. In the following we prove some hybrid coincidence points results in same settings. Examples are furnished to validate the results. As an application we consider the following differential inclusion problem for conduction in the metals with a closed and bounded set of source functions given as follows:

$$\begin{aligned} \frac{\partial^2\mu(\varkappa,t)}{\partial\varkappa^2}-\frac{\partial\mu(\varkappa,t)}{\partial t} &\in F(\varkappa,t,\mu,\mu_\varkappa), \text{ for } -\infty<\varkappa<\infty,\ 0<t<J, \\ \text{with } \mu(\varkappa,0) &= \tau(\varkappa),\ -\infty<\varkappa<\infty, \end{aligned}$$

The result that describes the conditions for the existence of the solution the above differential inclusion has been proved. We remark that our fixed point results are not consequences of classical results as mentioned in [18].

2. Preliminaries

In this section we recall some basic concepts, definitions and results mostly from [19, 28], which will be useful in the proofs of our main results.

Definition 2.1. *Let W be a non-empty set, a real valued function $G : W\times W\times W \to \mathbb{R}^+$ satisfying the following properties,*
(1) $G(\mu,\upsilon,w)=0$ *iff* $\mu=\upsilon=w$,
(2) $0<G(\mu,\mu,\upsilon)$, $\forall$ *distinct* $\mu,\upsilon\in W$,
(3) $G(\upsilon,\upsilon,w)\le G(\mu,\upsilon,w)$ *for all* $\mu,\upsilon,w\in W$ *with* $\upsilon\ne w$,
(4) $G(\mu,\upsilon,w)=G(\mu,w,\upsilon)=G(\upsilon,w,\mu)=\cdots$, *i.e., preserves symmetry,*
(5) $G(\mu,\upsilon,w)\le G(\mu,a,a)+G(a,\upsilon,w)\ \forall\ \mu,\upsilon,w,a\in W$,
is called a generalized metric, or, more specifically a G-metric on W, and the pair (W,G) is called a G-metric space. A G-metric is said to be symmetric if $G(\varkappa,\gamma,\gamma)=G(\gamma,\varkappa,\varkappa)$ for all $\varkappa,\gamma,\varsigma\in W$.

Definition 2.2. *Let (W,G) be a G-metric space, a sequence $\{\varkappa_p\}$ in W is said to be a G-convergent sequence if, for any $\varepsilon>0$, there exist an $\varkappa\in W$ and $N_0\in\mathbb{N}$ such that $G(\varkappa,\varkappa_p,\varkappa_q)<\varepsilon$, for all $p,q\ge N_0$, and a G-Cauchy sequence if, for any $\varepsilon>0$, there exists $N_0\in\mathbb{N}$ such that $G(\varkappa_p,\varkappa_q,\varkappa_l)<\varepsilon$, for all $p,q,l\ge N_0$. The generalized metric space (W,G) is called complete if every G-Cauchy sequence is G-convergent.*

The Hausdorff G-distance on closed and bounded subsets of W (denoted by $CB(W)$) is listed as;

$$H^G(H,K,S)=\max\{\sup_{\varkappa\in H}G(\varkappa,K,S),\sup_{\varkappa\in K}G(\varkappa,S,H),\sup_{\varkappa\in S}G(\varkappa,H,K)\}$$

where

$$\begin{gathered} G(\varkappa,K,S)=d_G(\varkappa,K)+d_G(K,S)+d_G(\varkappa,S) \\ d_G(\varkappa,K)=\inf\{d_G(\varkappa,\gamma):\gamma\in K\} \end{gathered}$$

$$d_G(K,S) = \inf\{d_G(a,b) : a \in K, b \in S\}.$$

Recall that

$$G(\varkappa,\gamma,S) = \inf\{G(\varkappa,\gamma,\varsigma) : \varsigma \in S\}.$$

A point $\varkappa \in W$ is called a fixed point of $F : W \to 2^W$, if $\varkappa \in F\varkappa$.

Remark 2.3. *Let W be a G-metric space, $\varkappa \in W$ and $K \subseteq W$. For each $\gamma \in K$,*

$$G(\varkappa, K, K) \leq 6G(\varkappa,\gamma,\gamma).$$

3. Main results

This section is divided into two subsections. In the first subsection, we initiate some new fixed point results for multivalued operators with contractive conditions of rational and non-rational types. The second subsection contains some new coincidence point results for hybrid mappings, satisfying new generalized contractive conditions.

3.1. **Fixed Point Results.** We present our first main results for finding fixed point of a multivalued mapping with rational type of contractive conditions.

Theorem 3.2. *Let W be a G-complete G-metric space and J be a mapping from W to $CB(W)$. If J satisfies the following condition:*

$$H^G(J\varkappa, J\gamma, J\varsigma) \leq a\left(\frac{G(J\varkappa,\gamma,\varsigma) + G(\varkappa, J\varkappa, J\varkappa)}{6G(\varkappa,\gamma,\varsigma)+1}\right)G(\varkappa,\gamma,\varsigma) + bG\left(\gamma, J\gamma, J\gamma\right),$$

for all $\varkappa,\gamma,\varsigma \in W, a, b \in (0,1), a + 6b < 1$, then,
(a) J has at least one fixed point $\xi \in W$,
(b) for any $\varkappa \in W$, the sequence $\{J^p\varkappa\}$ G-converges to a fixed point of J.

Proof. Let $\varkappa_0$ be arbitrary and formulate the sequence $\{\varkappa_p\}$ such that $\varkappa_{p+1} \in J\varkappa_p$. For the triplet $(\varkappa_p, \varkappa_{p+1}, \varkappa_{p+1})$, we set

$$d_p = G\left(\varkappa_p, \varkappa_{p+1}, \varkappa_{p+1}\right).$$

Now consider

$$\begin{aligned}
0 &< d_1 = G\left(\varkappa_1, \varkappa_2, \varkappa_2\right) \leq H^G(J\varkappa_0, J\varkappa_1, J\varkappa_1) + a \\
&\leq a\left(\frac{G(J\varkappa_0,\varkappa_1,\varkappa_1) + G(\varkappa_0, J\varkappa_0, J\varkappa_0)}{6G(\varkappa_0,\varkappa_1,\varkappa_1)+1}\right)G(\varkappa_0,\varkappa_1,\varkappa_1) \\
&\quad + bG\left(\varkappa_1, J\varkappa_1, J\varkappa_1\right) + a
\end{aligned}$$

$$\begin{aligned}
&\leq a\left(\frac{G(\varkappa_1,\varkappa_1,\varkappa_1) + 6G(\varkappa_0,\varkappa_1,\varkappa_1)}{6G(\varkappa_0,\varkappa_1,\varkappa_1)+1}\right)G(\varkappa_0,\varkappa_1,\varkappa_1) + 6bG\left(\varkappa_1, \varkappa_2, \varkappa_2\right) + a \\
&= \left(\frac{a}{1-6b}\right)\left(\frac{6d_0}{6d_0+1}\right)d_0 + \left(\frac{a}{1-6b}\right).
\end{aligned}$$

Similarly

$$\begin{aligned}
0 \quad < \quad & d_2 = G\left(\varkappa_2, \varkappa_3, \varkappa_3\right) \leq H^G(J\varkappa_1, J\varkappa_2, J\varkappa_2) + \frac{a^2}{1-6b} \\
\leq \quad & a\left(\frac{G(J\varkappa_1, \varkappa_2, \varkappa_2) + G(\varkappa_1, J\varkappa_1, J\varkappa_1)}{6G(\varkappa_1, \varkappa_2, \varkappa_2)+1}\right) G(\varkappa_1, \varkappa_2, \varkappa_2) \\
& +bG\left(\varkappa_2, J\varkappa_2, J\varkappa_2\right) + \frac{a^2}{1-6b}
\end{aligned}$$

$$\begin{aligned}
\leq \quad & a\left(\frac{G(\varkappa_2, \varkappa_2, \varkappa_2) + 6G(\varkappa_1, \varkappa_2, \varkappa_2)}{6G(\varkappa_1, \varkappa_2, \varkappa_2)+1}\right) G(\varkappa_1, \varkappa_2, \varkappa_2) \\
& +6bG\left(\varkappa_2, \varkappa_3, \varkappa_3\right) + \frac{a^2}{1-6b}
\end{aligned}$$

$$\begin{aligned}
= \quad & \left(\frac{a}{1-6b}\right)\left(\frac{6d_1}{6d_1+1}\right) d_1 + \left(\frac{a}{1-6b}\right)^2 \\
\leq \quad & \left(\frac{a}{1-6b}\right)^2 \left(\frac{6d_1}{6d_1+1}\right)\left(\frac{6d_0}{6d_0+1}\right) d_0 + \left(\frac{a}{1-6b}\right)^2 \left(\frac{6d_1}{6d_1+1}\right) \\
& + \left(\frac{a}{1-6b}\right)^2,
\end{aligned}$$

Assuming, $l = \frac{a}{1-6b}, \rho_p = \frac{6d_p}{6d_p+1}$ we get

$$\begin{aligned}
d_1 \quad & \leq \quad l\rho_0 d_0 + l, \\
d_2 \quad & \leq \quad l\rho_1 d_1 + l^2 \\
& \leq \quad l^2\rho_1\rho_0 d_0 + l^2\rho_1 + l^2,
\end{aligned}$$

$$\begin{aligned}
d_3 \quad & \leq \quad l\rho_2 d_2 + l^3 \\
& \leq \quad l^3\rho_2\rho_1\rho_0 d_0 + l^3\rho_2\rho_1 + l^3\rho_2 + l^3, \\
d_p \quad & \leq \quad l^p\rho_{p-1}...\rho_1\rho_0 d_0 + l^p\rho_{p-1}...\rho_1 + l^p\rho_{p-1}...\rho_2 + ... \\
& \quad\quad + l^p\rho_{p-1}\rho_{p-2} + l^p\rho_{p-1} + l^p.
\end{aligned}$$

Since $a + 6b < 1$, so $l < 1$ and $l^p \to 0$ as $p \to \infty$, so we have

$$\lim_{p\to\infty} d_p = 0.$$

For any two natural numbers q and p, with $q > p$, and by using triangular inequality we have

$$\begin{aligned} G(\varkappa_p, \varkappa_q, \varkappa_q) &\leq G(\varkappa_p, \varkappa_{p+1}, \varkappa_{p+1}) + G(\varkappa_{p+1}, \varkappa_q, \varkappa_q) \\ &\leq G(\varkappa_p, \varkappa_{p+1}, \varkappa_{p+1}) + ... + G(\varkappa_{q-1}, \varkappa_q, \varkappa_q) \\ &\leq \sum_{j=0}^{q-p-1} G(\varkappa_{p+j}, \varkappa_{p+j+1}, \varkappa_{p+j+1}) \\ &= \sum_{j=0}^{q-p-1} d_{p+j}, \end{aligned}$$

which gives $G(\varkappa_p, \varkappa_q, \varkappa_q) \to 0$ as $q, p \to \infty$. That is, $\{\varkappa_p\}$ is a G-Cauchy sequence. So G-completeness ensures the existence of some $\omega \in W$ such that $\varkappa_p \to \omega$.
Using definition of H^G, consider

$$\begin{aligned} G(\varkappa_{p+1}, J\omega, J\omega) &\leq H^G(J\varkappa_p, J\omega, J\omega) \\ &\leq a\left(\frac{G(J\varkappa_p, \omega, \omega) + G(\varkappa_p, J\varkappa_p, J\varkappa_p)}{6G(\varkappa_p, \omega, \omega) + 1}\right) G(\varkappa_p, \omega, \omega) \\ &\quad +bG(\omega, J\omega, J\omega). \end{aligned}$$

Taking limit on the both sides of above inequality, we get

$$(1-b)\, G(\omega, J\omega, J\omega) \leq 0.$$

Since $0 < b < 1$, therefore $G(\omega, J\omega, J\omega) = 0$, that is $\omega \in J\omega$. This proves the result. □

Example 3.3. *Let $W = \{0, 1, 2\}$. Define a mapping $J : W \to CB(W)$ by $J0 = \{0\} = J1, J2 = \{1\}$. Define a G-metric on W by*

$$\begin{aligned} G(\varkappa, \gamma, \varsigma) &= 1000, \varkappa \neq \gamma \neq \varsigma, \\ G(0, 1, 1) &= 0.008, G(0, 0, 1) = 0.01, \\ G(\varkappa, \varkappa, \varkappa) &= 0, G(\varkappa, \gamma, \varsigma) = 600, \ \textit{otherwise}. \end{aligned}$$

Then for $a = \frac{9}{10}, b = \frac{1}{100}$, all requirements of Theorem 1 are fulfilled and 0 is the fixed point of mapping J.

In the next result, we use an other new contractive condition to find fixed point of a multivalued mapping.

Theorem 3.4. *Let (W, G) be a G-complete G-metric space where J is a mapping from W to $CB(W)$. If J satisfies the following condition:*

$$\begin{aligned}
H^G(J\varkappa, J\gamma, J\varsigma) \quad \leq \quad & a_1 G(\varkappa, \gamma, \varsigma) + a_2 \begin{bmatrix} G(\varkappa, J\varkappa, J\varkappa)+ \\ G(\gamma, J\gamma, J\gamma)+ \\ G(\varsigma, J\varsigma, J\varsigma) \end{bmatrix} \\
& + a_3 \begin{bmatrix} G(J\varkappa, \gamma, \varsigma)+ \\ G(\varkappa, J\gamma, \varsigma)+ \\ G(\varkappa, \gamma, J\varsigma) \end{bmatrix} \frac{G(J\varkappa, \gamma, \varsigma)}{1 + G(\varkappa, \gamma, \varsigma)} \\
& + a_4 \min \begin{Bmatrix} G(J\varkappa, \gamma, \varsigma), \\ G(\gamma, J\gamma, J\gamma), \\ G(\varsigma, J\varsigma, J\varsigma) \end{Bmatrix} \frac{[1 + G(\varkappa, J\varkappa, J\varkappa)]}{1 + G(\varkappa, \gamma, \varsigma)} \\
& + a_5 \begin{bmatrix} 1 + G(\varkappa, J\gamma, \varsigma)+ \\ G(\varkappa, \gamma, J\varsigma) \end{bmatrix} \frac{G(J\varkappa, \gamma, \varsigma)}{1 + G(\varkappa, \gamma, \varsigma)} \\
& + a_6 \begin{bmatrix} 1 + G(\varkappa, J\varkappa, \varsigma)+ \\ G(J\varkappa, \gamma, \varsigma) \end{bmatrix} \frac{G(\varkappa, \gamma, \varsigma)}{1 + G(\varkappa, \gamma, \varsigma)} \\
& + a_7 G(\gamma, J\gamma, J\varsigma)
\end{aligned}$$

for all $\varkappa, \gamma, \varsigma \in W$ where $a_i, i = 1, \cdots, 7$, are nonnegative scalars such that

$$\begin{aligned}
\alpha &= a_1 + 6a_2 + a_6 < 1, \\
\beta &= ka_2 + la_7 < 1, k, l \in \mathbb{N} \\
\alpha + \beta &< 1,
\end{aligned}$$

then, J has a fixed point in W.

Proof. Let $\varkappa_0$ be arbitrary and formulate the sequence $\{\varkappa_p\}$ such that $\varkappa_{p+1} \in J\varkappa_p$. For the triplet $(\varkappa_p, \varkappa_{p+1}, \varkappa_{p+1})$, we set

$$d_p = G(\varkappa_p, \varkappa_{p+1}, \varkappa_{p+1}).$$

Now consider

$$
\begin{aligned}
0 &< d_1 = G\left(\varkappa_1, \varkappa_2, \varkappa_2\right) \leq H^G(J\varkappa_0, J\varkappa_1, J\varkappa_1) + \alpha \\
&\leq a_1 G(\varkappa_0, \varkappa_1, \varkappa_1) + a_2 \left[\begin{array}{c} G(\varkappa_0, J\varkappa_0, J\varkappa_0) + \\ G(\varkappa_1, J\varkappa_1, J\varkappa_1) + G(\varkappa_1, J\varkappa_1, J\varkappa_1) \end{array} \right] \\
&\quad + a_3 \left[\begin{array}{c} G(J\varkappa_0, \varkappa_1, \varkappa_1) + \\ G(\varkappa_0, J\varkappa_1, \varkappa_1) + \\ G(\varkappa_0, \varkappa_1, J\varkappa_1) \end{array} \right] \frac{G(J\varkappa_0, \varkappa_1, \varkappa_1)}{1 + G(\varkappa_0, \gamma, \varsigma)} \\
&\quad + a_4 \min \left\{ \begin{array}{c} G(J\varkappa_0, \varkappa_1, \varkappa_1), \\ G(\varkappa_1, J\varkappa_1, J\varkappa_1), \\ G(\varkappa_1, J\varkappa_1, J\varkappa_1) \end{array} \right\} \frac{[1 + G(\varkappa_0, J\varkappa_0, J\varkappa_0)]}{1 + G(\varkappa_0, \varkappa_1, \varkappa_1)} \\
&\quad + a_5 \left[\begin{array}{c} 1 + G(\varkappa_0, J\varkappa_1, \varkappa_1) + \\ G(\varkappa_0, \varkappa_1, J\varkappa_1) \end{array} \right] \frac{G(J\varkappa_0, \varkappa_1, \varkappa_1)}{1 + G(\varkappa_0, \gamma, \varsigma)} \\
&\quad + a_6 \left[\begin{array}{c} 1 + G(\varkappa_0, J\varkappa_0, \varkappa_1) + \\ G(J\varkappa_0, \varkappa_1, \varkappa_1) \end{array} \right] \frac{G(\varkappa_0, \varkappa_1, \varkappa_1)}{1 + G(\varkappa_0, \varkappa_1, \varkappa_1)} \\
&\quad + a_7 G(\varkappa_1, J\varkappa_1, J\varkappa_1) + \alpha,
\end{aligned}
$$

which gives

$$d_1 \leq a_1 d_0 + a_2 [6d_0 + 12d_1] + a_6 d_0 + a_7 \left[6d_1\right] + \alpha$$

that is

$$d_1 \leq \left(\frac{\alpha}{1-\beta}\right) d_0 + \left(\frac{\alpha}{1-\beta}\right).$$

Similarly

$$0 < d_2 = G\left(\varkappa_2, \varkappa_3, \varkappa_3\right) \leq H^G(J\varkappa_1, J\varkappa_2, J\varkappa_2) + \alpha,$$

which implies

$$
\begin{aligned}
d_2 &\leq \left(\frac{\alpha}{1-\beta}\right) d_1 + \left(\frac{\alpha}{1-\beta}\right) \\
&\leq \left(\frac{\alpha}{1-\beta}\right)^2 d_0 + \left(\frac{\alpha}{1-\beta}\right)^2 + \left(\frac{\alpha}{1-\beta}\right).
\end{aligned}
$$

Continuing this way we get

$$d_p \leq \left(\frac{\alpha}{1-\beta}\right)^p d_0 + \left(\frac{\alpha}{1-\beta}\right)^p + \ldots + \left(\frac{\alpha}{1-\beta}\right)^2 + \left(\frac{\alpha}{1-\beta}\right).$$

Since $\alpha + \beta < 1$, so $\left(\frac{\alpha}{1-\beta}\right) < 1$, and letting $p \to \infty$, we get

$$\lim_{p\to\infty} d_p = 0.$$

Applying the same steps as in the Theorem 1, it can be shown that $\{\varkappa_p\}$ is a G-Cauchy sequence. The G-completeness ensures the existence of some $\omega \in W$ such that $\varkappa_p \to \omega$.

Using definition of H^G we consider,

$$\begin{aligned}
G\left(\varkappa_{p+1}, J\omega, J\omega\right) &\leq H^G(J\varkappa_p, J\omega, J\omega) \\
&\leq a_1 G(\varkappa_p, \omega, \omega) + a_2 \begin{bmatrix} G(\varkappa_p, J\varkappa_p, J\varkappa_p)+ \\ G(\omega, J\omega, J\omega)+ \\ G(\omega, J\omega, J\omega) \end{bmatrix} \\
&\quad + a_3 \begin{bmatrix} G(J\varkappa_p, \omega, \omega)+ \\ G(\varkappa_p, J\omega, \omega)+ \\ G(\varkappa_p, \omega, J\omega) \end{bmatrix} \frac{G(J\varkappa_p, \omega, \omega)}{1 + G(\varkappa_p, \omega, \omega)} \\
&\quad + a_4 \min \left\{ \begin{array}{c} G(J\varkappa_p, \omega, \omega), \\ G(\omega, J\omega, J\omega), \\ G(\omega, J\omega, J\omega) \end{array} \right\} \frac{[1 + G(\varkappa_p, J\varkappa_p, J\varkappa_p)]}{1 + G(\varkappa_p, \omega, \omega)} \\
&\quad + a_5 \begin{bmatrix} 1 + G(\varkappa_p, J\omega, \omega)+ \\ G(\varkappa_p, \omega, J\omega) \end{bmatrix} \frac{G(J\varkappa_p, \omega, \omega)}{1 + G(\varkappa_p, \omega, \omega)} \\
&\quad + a_6 [1 + G(\varkappa_p, J\varkappa_p, \omega) + G(J\varkappa_p, \omega, \omega)] \frac{G(\varkappa_p, \omega, \omega)}{1 + G(\varkappa_p, \omega, \omega)} \\
&\quad + a_7 G(\omega, J\omega, J\omega).
\end{aligned}$$

Applying limit $p \to \infty$, we have

$$G(\omega, J\omega, J\omega) \leq 3a_2 G(\omega, J\omega, J\omega) + a_7 G(\omega, J\omega, J\omega),$$

that is

$$(1 - 3a_2 - a_7)\, G(\omega, J\omega, J\omega) \leq 0.$$

As $3a_2 + a_7 < 1$ and $G(\omega, J\omega, J\omega) = 0$ which implies $\omega \in J\omega$. This proves the theorem. $\square$

Example 3.5. *Let $W = \{0, 1, 2\}$. Define a mapping $J : W \to CB(W)$ by $J0 = \{0\} = J1, J2 = \{1\}$. Define a G-metric on W by*

$$\begin{aligned}
G(\varkappa, \gamma, \varsigma) &= 1000, \varkappa \neq \gamma \neq \varsigma, \\
G(0, 1, 1) &= 0.008, G\left(0, 0, 1\right) = 0.01, \\
G(\varkappa, \varkappa, \varkappa) &= 0, G(\varkappa, \gamma, \varsigma) = 600, \textit{ otherwise}.
\end{aligned}$$

Then for $a_1 = \frac{9}{10}, a_2 = \frac{1}{100}, a_3 = 2, a_4 = 4, a_5 = 5, a_6 = \frac{1}{1000}, a_7 = \frac{1}{10000}$, all requirements of Theorem 2, are fulfilled and 0 is the fixed point of mapping J.

3.6. **Coincidence Point Theorems.** Now we present new coincidence point theorems for hybrid mappings with a generalize contractive condition.

Theorem 3.7. *Let W be a G-metric space. Assuming that $g : W \to W, Q, J : W \to CB(W)$ are satisfing*

$$\begin{aligned}
H^G(J\varkappa, Q\gamma, Q\varsigma) &\leq aG(g\varkappa, g\gamma, g\varsigma) + bG(g\varkappa, J\varkappa, J\varkappa) \\
&\quad + cG(g\gamma, Q\gamma, Q\gamma) + dG(g\varsigma, Q\varsigma, Q\varsigma),
\end{aligned} \tag{3. 1}$$

and

$$H^G(Q\varkappa, J\gamma, J\varsigma) \leq aG(g\varkappa, g\gamma, g\varsigma) + bG(g\varkappa, Q\varkappa, Q\varkappa) + cG(g\gamma, J\gamma, J\gamma) + dG(g\varsigma, J\varsigma, J\varsigma) \quad (3.\ 2)$$

for all $\varkappa, \gamma, \varsigma \in W$, *where* $a, b, c, d \in [0, 1)$, *and* $a + 6b + 6c + 6d < 1, c + d < 1$. *If*
(i) $J(W) \subseteq g(W), Q(W) \subseteq g(W)$,
(ii) $g(W)$ *is a G-complete subspace of* W,
then, there exists ω *in* W *such that* $g\omega \in J\omega \cap Q\omega$.

Proof. For arbitrary $\varkappa_0 \in W$, we consider a sequence $\{g\varkappa_p\}$ in $g(W)$ such that

$$g\varkappa_{2p} \in J\varkappa_{2p-1}, g\varkappa_{2p+1} \in Q\varkappa_{2p}.$$

Let

$$d_p = G\left(g\varkappa_p, g\varkappa_{p+1}, g\varkappa_{p+1}\right),$$

then for $g\varkappa_1, g\varkappa_2$ in the sequence, by using inequality (3.2) we get

$$\begin{aligned} 0 &< d_1 = G\left(g\varkappa_1, g\varkappa_2, g\varkappa_2\right) \leq H^G(Q\varkappa_0, J\varkappa_1, J\varkappa_1) + (a + 6b) \\ &\leq aG(g\varkappa_0, g\varkappa_1, g\varkappa_1) + bG(g\varkappa_0, Q\varkappa_0, Q\varkappa_0) \\ &\quad + cG(g\varkappa_1, J\varkappa_1, J\varkappa_1) + dG(g\varkappa_1, J\varkappa_1, J\varkappa_1) + (a + 6b) \\ &\leq ad_0 + 6bd_0 + 6cd_1 + 6dd_1 + (a + 6b), \end{aligned}$$

which gives

$$d_1 \leq \left(\frac{a + 6b}{1 - 6c - 6d}\right) d_0 + \left(\frac{a + 6b}{1 - 6c - 6d}\right).$$

Now for $g\varkappa_2, g\varkappa_3$ in the sequence and using inequality (3.1) we get

$$\begin{aligned} 0 &< d_2 = G\left(g\varkappa_2, g\varkappa_3, g\varkappa_3\right) \leq H^G(J\varkappa_1, Q\varkappa_2, Q\varkappa_2) + (a + 6b) \\ &\leq aG(g\varkappa_1, g\varkappa_2, g\varkappa_2) + bG(g\varkappa_1, J\varkappa_1, J\varkappa_1) \\ &\quad + cG(g\varkappa_2, Q\varkappa_2, Q\varkappa_2) + dG(g\varkappa_2, Q\varkappa_2, Q\varkappa_2) + (a + 6b) \\ &\leq ad_1 + 6bd_1 + 6cd_2 + 6dd_2 + (a + 6b), \end{aligned}$$

which gives

$$\begin{aligned} d_2 &\leq \left(\frac{a + 6b}{1 - 6c - 6d}\right) d_1 + \left(\frac{a + 6b}{1 - 6c - 6d}\right) \\ &\leq \left(\frac{a + 6b}{1 - 6c - 6d}\right)^2 d_0 + \left(\frac{a + 6b}{1 - 6c - 6d}\right)^2 + \left(\frac{a + 6b}{1 - 6c - 6d}\right). \end{aligned}$$

Applying similar steps as we perform in the proof of Theorem 1, it can be shown that $\{g\varkappa_p\}$ is a G-Cauchy sequence. The G-completeness ensures the existence of some $g\omega \in g(W)$ such that $g\varkappa_p \to g\omega$.
Now consider,

$$\begin{aligned} G\left(g\varkappa_{2p+1}, J\omega, J\omega\right) &\leq H^G(Q\varkappa_{2p}, J\omega, J\omega) \\ &\leq aG(g\varkappa_{2p}, g\omega, g\omega) + bG(g\varkappa_{2p}, Q\varkappa_{2p}, Q\varkappa_{2p}) \\ &\quad + cG(g\omega, J\omega, J\omega) + dG(g\omega, J\omega, J\omega) \end{aligned}$$

As $p \to \infty$, we get
$$(1-c-d)\, G(g\omega, J\omega, J\omega) \leq 0.$$
Since $c+d<1$, therefore $G(g\omega, J\omega, J\omega)=0$ which implies $g\omega \in J\omega$.
Now consider,
$$\begin{aligned} G\left(g\varkappa_{2p}, Q\omega, Q\omega\right) &\leq H^G(J\varkappa_{2p-1}, Q\omega, Q\omega) \\ &\leq aG(g\varkappa_{2p-1}, g\omega, g\omega) + bG(g\varkappa_{2p-1}, J\varkappa_{2p-1}, J\varkappa_{2p-1}) \\ &\quad + cG(g\omega, Q\omega, Q\omega) + dG(g\omega, Q\omega, Q\omega) \end{aligned}$$
As $p \to \infty$, we get
$$(1-c-d)\, G(g\omega, Q\omega, Q\omega) \leq 0.$$
Again as, $c+d<1$ therefore $G(g\omega, Q\omega, Q\omega)=0$ which implies $g\omega \in Q\omega$. □

Example 3.8. *Let $W=[0,1]$, define a mapping $g: W \to W$ and $J, Q: W \to CB(W)$ by $g\varkappa = \frac{3\varkappa}{4}$,*
$$J\varkappa = \left[0, \frac{\varkappa}{25}\right] \quad \text{and} \quad Q\varkappa = \left[0, \frac{\varkappa}{20}\right]$$
respectively. Then for $a=\frac{1}{11}, b=\frac{1}{24}, c=\frac{1}{50}, d=\frac{2}{35}$,all requirements of Theorem 3 are fulfilled and 0 is the coincidence point of mappings Q, J and g.

Corollary 3.9. *Let (W,G) be a G-complete G-metric space. Assume that $Q, J: W \to CB(W)$ satisfy*
$$\begin{aligned} H^G(J\varkappa, Q\gamma, Q\varsigma) &\leq aG(\varkappa,\gamma,\varsigma) + bG(\varkappa, J\varkappa, J\varkappa) \\ &\quad + cG(\gamma, Q\gamma, Q\gamma) + dG(\varsigma, Q\varsigma, Q\varsigma), \end{aligned}$$
and
$$\begin{aligned} H^G(Q\varkappa, J\gamma, J\varsigma) &\leq aG(\varkappa,\gamma,\varsigma) + bG(\varkappa, Q\varkappa, Q\varkappa) \\ &\quad + cG(\gamma, J\gamma, J\gamma) + dG(\varsigma, J\varsigma, J\varsigma) \end{aligned}$$
for all $\varkappa,\gamma,\varsigma \in W$, where $a,b,c,d \in [0,1)$, and $a+6b+6c+6d<1, c+d<1$. Then there exists ω in W such that $\omega \in J\omega \cap Q\omega$.

Proof. By assuming $g=I$ in the Theorem 3, we obtain the required result. □

Theorem 3.10. *Let (W,G) be a G-complete G-metric space. Assume that $J: W \to CB(W)$ satisfy*
$$H^G(J\varkappa, J\gamma, J\varsigma) \leq \frac{a}{6} G(\varkappa,\gamma,\varsigma), \tag{3.3}$$
for all $\varkappa,\gamma,\varsigma \in W$, where $a \in [0,1)$. Then there exists ω in W such that $\omega \in J\omega$.

Proof. For arbitrary $\varkappa_0 \in W$, $Jx_0 \in CB(W)$ and $Jx_0 \neq \varphi$ implies the existence of an element $x_1 \in Jx_0$. In continuation a sequence $\{x_n\}$ in W such that $\varkappa_{n+1} \in J\varkappa_n$.
By using condition (3.3) we get
$$\begin{aligned} G\left(\varkappa_1, \varkappa_2, \varkappa_2\right) &\leq H^G(J\varkappa_0, J\varkappa_1, J\varkappa_1) + \frac{a}{6} \\ &\leq \frac{a}{6} G(\varkappa_0, \varkappa_1, \varkappa_1) + \frac{a}{6}. \end{aligned}$$

$$\begin{aligned} G\left(\varkappa_2, \varkappa_3, \varkappa_3\right) &\leq H^G(J\varkappa_1, J\varkappa_2, J\varkappa_2) + \frac{a^2}{6^2} \\ &\leq \frac{a}{6}G(\varkappa_1, \varkappa_2, \varkappa_2) + \frac{a^2}{6^2} \\ &\leq \frac{a^2}{6^2}G(\varkappa_0, \varkappa_1, \varkappa_1) + \frac{2a^2}{6^2}. \end{aligned}$$

Similarily, we have

$$G\left(\varkappa_n, \varkappa_{n+1}, \varkappa_{n+1}\right) \leq \frac{a^n}{6^n}G(\varkappa_0, \varkappa_1, \varkappa_1) + \frac{na^n}{6^n}.$$

Consider for $m > n$ and using inequality (3.4) we get

$$\begin{aligned} G\left(\varkappa_n, \varkappa_m, \varkappa_m\right) &\leq G\left(\varkappa_n, \varkappa_{n+1}, \varkappa_{n+1}\right) + G\left(\varkappa_{n+1}, \varkappa_{n+2}, \varkappa_{n+2}\right) + ... + G\left(\varkappa_{m-1}, \varkappa_m, \varkappa_m\right) \\ &\leq \frac{\frac{a^n}{6^n}}{1-\frac{a}{6}}G(\varkappa_0, \varkappa_1, \varkappa_1) + \sum_{i=n}^{m-1}\frac{ia^i}{6^i}. \end{aligned}$$

Applying limit $m, n \to \infty$, the sequence $\{x_n\}$ is proved to be a G-Cauchy sequence. The G-completeness ensures the existence of some $\omega \in W$ such that $\varkappa_n \to \omega$.
Now consider,

$$\begin{aligned} G\left(\varkappa_{n+1}, J\omega, J\omega\right) &\leq H^G(J\varkappa_n, J\omega, J\omega) \\ &\leq \frac{a}{6}G(\varkappa_n, \omega, \omega) \end{aligned}$$

As $n \to \infty$, we get

$$G(\omega, J\omega, J\omega) = 0,$$

which implies $\omega \in J\omega$. □

4. Application

Differential inclusions arise in many physical problems, such as arise for Amonton-Coulomb friction model in mechanical systems, in the theory of differential games and ideal switches in power electronics. Some certain differential inclusions also originate at the groundwork of non-smooth dynamical system analysis which is used in the analog study of switching electrical circuits using idealized component equations(see [1, 21, 39, 40, 43, 44, 47, 48]). In the following we present a differential inclusion with a compact set of source functions, i.e., many source functions can be taken, this problem is very useful in the theory of conduction in the metals. The problem is given as;

$$\frac{\partial^2\mu\left(\varkappa,\gamma\right)}{\partial\varkappa^2} - \frac{\partial\mu\left(\varkappa,\gamma\right)}{\partial\gamma} \in F\left(\varkappa,\gamma,\mu,\mu_\varkappa\right), \text{ for } -\infty < \varkappa < \infty,\ 0 < \gamma < T, \quad (4.\ 3)$$

with $\mu(\varkappa, 0) = \tau\left(\varkappa\right),\ -\infty < \varkappa < \infty$, for $(\varkappa, \gamma, \mu, \mu_\varkappa) \in \Omega_{\mathbb{R},T} \times \mathbb{R}^p \times \mathbb{R}^p$, where $\Omega_{\mathbb{R},B} = \mathbb{R} \times \Omega_T$ such that $\Omega_T = [0, T]$ and $\tau\left(\varkappa\right)$ and $\acute{\tau}\left(\varkappa\right)$ are bounded and $\tau\left(\varkappa\right)$ is assumed to be continuously differentiable.

We use one of our results to find the existence of the solution of above differential inclusion, in the following settings.

Let $\Sigma = C\left(\Omega_{\mathbb{R},T}, \mathbb{R}^p\right)$, and define a G-metric on Σ as follows:

$$\begin{aligned}
\left(G\left(w_1, w_2, w_3\right)\right)(\varkappa, \gamma) \quad = \quad & \sup_{(\varkappa,\gamma)\in\Omega_{\mathbb{R},T}} \left\| w_1(\varkappa, \gamma) - w_2(\varkappa, \gamma) \right\| + \\
& \sup_{(\varkappa,\gamma)\in\Omega_{\mathbb{R},T}} \left\| \frac{\partial w_1(\varkappa, \gamma)}{\partial \varkappa} - \frac{\partial w_2(\varkappa, \gamma)}{\partial \varkappa} \right\| + \\
& \sup_{(\varkappa,\gamma)\in\Omega_{\mathbb{R},T}} \left\| w_2(\varkappa, \gamma) - w_3(\varkappa, \gamma) \right\| + \\
& \sup_{(\varkappa,\gamma)\in\Omega_{\mathbb{R},T}} \left\| \frac{\partial w_2(\varkappa, \gamma)}{\partial \varkappa} - \frac{\partial w_3(\varkappa, \gamma)}{\partial \varkappa} \right\| + \\
& \sup_{(\varkappa,\gamma)\in\Omega_{\mathbb{R},T}} \left\| w_1(\varkappa, \gamma) - w_3(\varkappa, \gamma) \right\| + \\
& \sup_{(\varkappa,\gamma)\in\Omega_{\mathbb{R},T}} \left\| \frac{\partial w_1(\varkappa, \gamma)}{\partial \varkappa} - \frac{\partial w_3(\varkappa, \gamma)}{\partial \varkappa} \right\|,
\end{aligned}$$

for $w_1, w_2, w_3 \in \Sigma$. Then Σ is a complete G-metric space. Define a partial order $\leq$ on Σ as

$$\begin{aligned}
p, q \quad &\in \quad \Sigma,\ \ p \leq q \quad \text{if and only if} \\
\left\| p(\varkappa, \gamma) \right\| \quad &\leq \quad \left\| q(\varkappa, \gamma) \right\| \ \text{ and } \left\| \frac{\partial p(\varkappa, \gamma)}{\partial \varkappa} \right\| \leq \left\| \frac{\partial q(\varkappa, \gamma)}{\partial \varkappa} \right\|.
\end{aligned}$$

Let $L = L^1\left(\Omega_{\mathbb{R},T}, \mathbb{R}^p\right)$ be the Banach space consisting of all measurable functions $\omega : \Omega_{\mathbb{R},T} \to \mathbb{R}^p$ which are also Lebesgue integrable with

$$\|\omega\|_{\mathcal{L}} = \left| \int_{-\infty}^{\infty} \int_0^T \omega(\varkappa, \gamma)\, d\gamma d\varkappa \right|, \ \text{ for } \ \omega \in \mathcal{L}.$$

Let $F : \Omega_{\mathbb{R},T} \times \mathbb{R}^p \times \mathbb{R}^p \to 2^{\mathbb{R}^p}$ be a set-valued mapping. For each s in the Banach space Σ, the collection of all selections of F is denoted and defined as:

$$\Theta_{F,s} = \left\{ \omega \in \mathcal{L} : \omega \in F\left(\varkappa, \gamma, s, s_\varkappa\right) \ a.e \ (\varkappa, \gamma) \in \Omega_{\mathbb{R},T} \right\}.$$

Now we assign a multivalued operator $\sigma : \Sigma \to 2^{\mathcal{L}}$ to the mapping F as

$$\sigma(s) = \left\{ \upsilon \in \mathcal{L} : \upsilon(\varkappa, \gamma) \in F\left(\varkappa, \gamma, s, s_\varkappa\right), (\varkappa, \gamma) \in \Omega_{\mathbb{R},T} \right\},$$

where σ represents the Niemytsky operator [4] associated with F.

For the purpose of our application theorem we consider a continuous mapping $\phi : L \to \Sigma$ defined as

$$\phi(s) = \int_0^\gamma \int_{-\infty}^{\infty} \xi(\vartheta, \varrho)\, s(\vartheta, \varrho)\, d\vartheta d\varrho.$$

In the next theorem, we use our Theorem 4 to obtain the solution of above differential inclusion 4.1.

Theorem 4.1. *Suppose the multivalued mapping* $F : \Omega_{\mathbb{R},T} \times \mathbb{R}^p \times \mathbb{R}^p \to CB\left(\mathbb{R}^p\right)$ *satisfies the subsequent conditions:*

(i) $F\left(\varkappa, \gamma, \mu, \mu_\varkappa\right)$ *is a closed and bounded subset for each element* $\left(\varkappa, \gamma, \mu, \mu_\varkappa\right) \in \Omega_{\mathbb{R},T} \times$

$\Sigma \times \Sigma$. *Also assume that* $\Theta_{F,s}$ *is non-empty for every* $s \in \Sigma$*;*
(ii) *for any* $\mu_1, \mu_2 \in \Sigma$, *if* $\mu_1 \lesssim \mu_2$ *then for each* $\upsilon_1(\varkappa, \gamma) \in F(\varkappa, \gamma, \mu_1, \mu_{1\varkappa})$ *there exists* $\upsilon_2(\varkappa, \gamma) \in F(\varkappa, \gamma, \mu_2, \mu_{2\varkappa})$ *such that*

$$\begin{aligned} |\nu_1(\varkappa,\gamma) - \nu_2(\varkappa,\gamma)| &\leq \frac{a}{\left(T + \frac{2}{\sqrt{\pi T}}\right)} \left[\frac{\begin{array}{c}(|\upsilon_1 - \mu_2| + |\upsilon_{1\varkappa} - \mu_{2\varkappa}|) + \\ (|\mu_1 - \upsilon_2| + |\mu_{1\varkappa} - \upsilon_{2\varkappa}|)\end{array}}{3\min(|\upsilon_1 - \mu_2| + |\upsilon_{1\varkappa} - \mu_{2\varkappa}|) + 1} \right] \\ &\quad \times 2(|\upsilon_1 - \mu_2| + |\upsilon_{1\varkappa} - \mu_{2\varkappa}|) \\ &\quad + 2\frac{b}{\left(T + \frac{2}{\sqrt{\pi T}}\right)} (|\mu_2 - \upsilon_2| + |\mu_{2\varkappa} - \upsilon_{2\varkappa}|), \\ \textit{for a.e } (\varkappa,\gamma) &\in \Omega_{\mathbb{R},T}, \textit{ and } a + 6b < 1, \end{aligned}$$

where $CB(\mathbb{R}^p)$ *is the set of closed and bounded subsets of* $\mathbb{R}^p$. *Then there exists a solution of the differential inclusion* (4.1).

Proof. Note that the problem 4.1 is equivalent to the following integral inclusion

$$\mu(\varkappa,\gamma) \in \left\{ \begin{array}{c} \kappa \in \Sigma : \kappa(\varkappa,\gamma) = \int\limits_{-\infty}^{\infty} \xi(\varkappa - \zeta, \gamma)\tau(\zeta) d\zeta + \\ \int\limits_{0}^{\gamma} \int\limits_{-\infty}^{\infty} \xi(\varkappa - \zeta, \gamma - \gamma)\omega(\zeta,\gamma)\, d\zeta d\gamma, \text{ for } \omega \in \Theta_{F,s} \end{array} \right\},$$

where $\xi(\varkappa,\gamma)$ represents the Green's function defined by

$$\xi(\varkappa,\gamma) = \frac{1}{\sqrt{4\pi\gamma}} \exp\left\{ \frac{-\varkappa^2}{4\gamma} \right\}.$$

Define the mappings $J : \Sigma \to 2^{\Sigma}$ by

$$(Js)(\varkappa,\gamma) = \left\{ \begin{array}{c} \kappa \in \Sigma : \kappa(\varkappa,\gamma) = \int\limits_{-\infty}^{\infty} \xi(\varkappa - \zeta, \gamma)\tau(\zeta) d\zeta + \\ \int\limits_{0}^{\gamma} \int\limits_{-\infty}^{\infty} \xi(\varkappa - \zeta, \gamma - \gamma)\omega(\zeta,\gamma)\, d\zeta d\gamma, \text{ for } \omega \in \Theta_{F,s} \end{array} \right\}.$$

Now we show that $(Js)(\varkappa,\gamma)$ is compact for each $s \in \Sigma$ and $\omega \in \Theta_{F,s}$. For this, it is enough to prove that $\phi \circ \Theta_{F,s}$ is compact. For this, let $\{\omega_p\}$ be a sequence in $\Theta_{F,s}$, then by the definition of $\Theta_{F,s}$

$$\omega_p \in \mathcal{L} \text{ and } \omega_p \in F(\varkappa,\gamma,s,s_\varkappa) \ a.e \ (\varkappa,\gamma) \in \Omega_{\mathbb{R},T}.$$

Since $F(\varkappa,\gamma,s,s_\varkappa)$ is compact so there exists $\upsilon(\varkappa,\gamma) \in F(\varkappa,\gamma,s,s_\varkappa)$ such that $\omega_p \to \nu$. By the continuity of ϕ, $\phi \circ \omega_p \to \phi \circ \nu$. Since $\nu \in L$ and $\upsilon(\varkappa,\gamma) \in F(\varkappa,\gamma,s,s_\varkappa)$ therefore $\phi \circ \nu \in \phi \circ \Theta_{F,s}$ which implies the compactness of $\phi \circ \Theta_{F,s}$.
Now suppose $w_1, w_2 \in \Sigma$ with $w_1 \leq w_2$ and $\kappa_1 \in Jw_1$, then there exists $\nu_1 \in \Theta_{F,w_1}$ such

that

$$\begin{aligned}\kappa_1(\varkappa,\gamma) &= \int_{-\infty}^{\infty} \xi(\varkappa-\zeta,\gamma)\tau(\zeta)d\zeta + \phi\left(\nu_1\left(\varkappa,\gamma\right)\right) \\ &= \int_{-\infty}^{\infty} \xi(\varkappa-\zeta,\gamma)\tau(\zeta)d\zeta + \int_{0}^{\gamma}\int_{-\infty}^{\infty} \xi\left(\varkappa-\vartheta,\gamma-\varrho\right)\nu_1\left(\vartheta,\varrho\right)d\varrho d\tau,\end{aligned}$$

for $(\varkappa,\gamma) \in \Omega_{\mathbb{R},T}$.

Now by the given assumption there exists a ς in $F\left(\varkappa,\gamma,w_2,w_{2\varkappa}\right)$ such that

$$\begin{aligned}\left|\nu_1\left(\varkappa,\gamma\right)-\varsigma\right| \leq{}& \frac{a}{\left(T+\frac{2}{\sqrt{\pi T}}\right)}\left[\frac{\left(\left|\upsilon_1-w_2\right|+\left|\upsilon_{1\varkappa}-w_{2\varkappa}\right|\right)+\left(\left|w_1-\upsilon_2\right|+\left|w_{1\varkappa}-\upsilon_{2\varkappa}\right|\right)}{3\min\left(\left|w_1-w_2\right|+\left|w_{1\varkappa}-w_{2\varkappa}\right|\right)+1}\right] \\ &\times 2\left(\left|w_1-w_2\right|+\left|w_{1\varkappa}-w_{2\varkappa}\right|\right)+2\frac{b}{\left(T+\frac{2}{\sqrt{\pi T}}\right)}\left(\left|w_2-\upsilon_2\right|+\left|w_{2\varkappa}-\upsilon_{2\varkappa}\right|\right).\end{aligned}$$

Define a multivalued mapping $\mho : \Omega_{\mathbb{R},T} \to \mathbb{R}^p$ by

$$\mho\left(\varkappa,\gamma\right)=\left\{\begin{array}{c}\varsigma\in\mathbb{R}^p : \left|\nu_1\left(\varkappa,\gamma\right)-\varsigma\right| \leq \frac{a}{T+\frac{2}{\sqrt{\pi T}}}\left[\frac{\left(\left|\upsilon_1-w_2\right|+\left|\upsilon_{1\varkappa}-w_{2\varkappa}\right|\right)+\left(\left|w_1-\upsilon_2\right|+\left|w_{1\varkappa}-\upsilon_{2\varkappa}\right|\right)}{3\min\left(\left|w_1-w_2\right|+\left|w_{1\varkappa}-w_{2\varkappa}\right|\right)+1}\right] \\ \times 2\left(\left|w_1-w_2\right|+\left|w_{1\varkappa}-w_{2\varkappa}\right|\right) \\ +2\frac{b}{T+\frac{2}{\sqrt{\pi T}}}\left(\left|w_2-\upsilon_2\right|+\left|w_{2\varkappa}-\upsilon_{2\varkappa}\right|\right)\end{array}\right\};$$

then the mapping $\Phi : \Omega_{\mathbb{R},T} \to \mathbb{R}^p$ defined by

$$\Phi\left(\varkappa,\gamma\right)=\mho\left(\varkappa,\gamma\right)\cap\Theta_{F,w_1}$$

is a measurable selection which has non-empty values [4]. So there exists ν_2 in Φ such that $\nu_2 \in F\left(\varkappa,\gamma,w_2,w_{2\varkappa}\right)$ and

$$\begin{aligned}\left|\nu_1\left(\varkappa,\gamma\right)-\nu_2\left(\varkappa,\gamma\right)\right| \leq{}& \frac{a}{\left(T+\frac{2}{\sqrt{\pi T}}\right)}\left[\frac{\begin{array}{c}\left(\left|\upsilon_1-w_2\right|+\left|\upsilon_{1\varkappa}-w_{2\varkappa}\right|\right)\\ +\left(\left|w_1-\upsilon_2\right|+\left|w_{1\varkappa}-\upsilon_{2\varkappa}\right|\right)\end{array}}{3\min\left(\left|w_1-w_2\right|+\left|w_{1\varkappa}-w_{2\varkappa}\right|\right)+1}\right] \\ &\times 2\left(\left|w_1-w_2\right|+\left|w_{1\varkappa}-w_{2\varkappa}\right|\right) \\ &+2\frac{b}{\left(T+\frac{2}{\sqrt{\pi T}}\right)}\left(\left|w_2-\upsilon_2\right|+\left|w_{2\varkappa}-\upsilon_{2\varkappa}\right|\right),\end{aligned}$$

for all $(\varkappa,\gamma) \in \Omega_{\mathbb{R},T}$.

For each $(\varkappa,\gamma)\in\Omega_{\mathbb{R},T}$, set

$$\begin{aligned}\kappa_2(\varkappa,\gamma) &= \int_{-\infty}^{\infty} \xi(\varkappa-\zeta,\gamma)\tau(\zeta)d\zeta + \phi\left(\nu_2\left(\varkappa,\gamma\right)\right) \\ &= \int_{-\infty}^{\infty} \xi(\varkappa-\zeta,\gamma)\tau(\zeta)d\zeta + \int_{0}^{\gamma}\int_{-\infty}^{\infty} \xi\left(\varkappa-\vartheta,\gamma-\varrho\right)\nu_2\left(\vartheta,\varrho\right)d\varrho d\tau.\end{aligned}$$

Then

$$
\begin{aligned}
\left|\kappa_1(\varkappa,\gamma)-\kappa_2(\varkappa,\gamma)\right| &\le \int_0^{\gamma}\int_{-\infty}^{\infty}\xi(\varkappa-\vartheta,\gamma-\varrho)\left|\nu_2(\vartheta,\varrho)-\nu_1(\vartheta,\varrho)\right| d\varrho d\tau \\
&\le \frac{a}{\left(T+\frac{2}{\sqrt{\pi T}}\right)}\left[\frac{\begin{array}{c}(|\upsilon_1-w_2|+|\upsilon_{1\varkappa}-w_{2\varkappa}|)+\\(|w_1-\upsilon_2|+|w_{1\varkappa}-\upsilon_{2\varkappa}|)\end{array}}{3\min(|w_1-w_2|+|w_{1\varkappa}-w_{2\varkappa}|)+1}\right] \\
&\quad \times 2(|w_1-w_2|+|w_{1\varkappa}-w_{2\varkappa}|) \\
&\quad +2\frac{b}{\left(T+\frac{2}{\sqrt{\pi T}}\right)}(|w_2-\upsilon_2|+|w_{2\varkappa}-\upsilon_{2\varkappa}|) \\
&\quad \times\int_0^{\gamma}\int_{-\infty}^{\infty}\xi(\varkappa-\vartheta,\gamma-\varrho)\, d\varrho d\tau \\
&\quad \text{Taking } \sup \text{ on both sides,} \\
&\le aT\left[\frac{G(Jw_1,w_2,w_2)+G(w_1,Jw_1,Jw_1)}{6G(w_1,w_2,w_2)+1}\right]G(w_1,w_2,w_2)+ \\
&\quad bTG(w_2,Jw_2,Jw_2),
\end{aligned}
$$

Now

$$
\begin{aligned}
\left|\kappa_{1\varkappa}(\varkappa,\gamma)-\kappa_{2\varkappa}(\varkappa,\gamma)\right| &\le \frac{a}{\left(T+\frac{2}{\sqrt{\pi T}}\right)}\left[\frac{\begin{array}{c}(|\upsilon_1-w_2|+|\upsilon_{1\varkappa}-w_{2\varkappa}|)+\\(|w_1-\upsilon_2|+|w_{1\varkappa}-\upsilon_{2\varkappa}|)\end{array}}{3\min(|w_1-w_2|+|w_{1\varkappa}-w_{2\varkappa}|)+1}\right] \\
&\quad \times 2(|w_1-w_2|+|w_{1\varkappa}-w_{2\varkappa}|) \\
&\quad +2\frac{b}{\left(T+\frac{2}{\sqrt{\pi T}}\right)}(|w_2-\upsilon_2|+|w_{2\varkappa}-\upsilon_{2\varkappa}|) \\
&\quad \times\int_0^{\gamma}\int_{-\infty}^{\infty}\left|\xi_{\varkappa}(\varkappa-\vartheta,\gamma-\varrho)\right| d\varrho d\tau \\
&\le \frac{a}{\left(T+\frac{2}{\sqrt{\pi T}}\right)}T\frac{2}{\sqrt{\pi T}}\left[\frac{\begin{array}{c}G(Jw_1,w_2,w_2)+\\G(w_1,Jw_1,Jw_1)\end{array}}{6G(w_1,w_2,w_2)+1}\right]G(w_1,w_2,w_2) \\
&\quad +\frac{b}{\left(T+\frac{2}{\sqrt{\pi T}}\right)}\frac{2}{\sqrt{\pi T}}TG(w_2,Jw_2,Jw_2).
\end{aligned}
$$

Thus we have

$$H^G(Jw_1, Jw_2, Jw_2) \leq a \left[\frac{\begin{array}{c} G(Jw_1, w_2, w_2) + \\ G(w_1, Jw_1, Jw_1) \end{array}}{6G(w_1, w_2, w_2) + 1} \right] G(w_1, w_2, w_2)$$
$$+ bG(w_2, Jw_2, Jw_2) \, ..$$

Thanks to main Theorem 1, to find $\mu \in \Sigma$ such that $\mu \in J\mu$, which is the solution of the Problem 4.1. □

5. Conclusion

In this article some fixed points and coincidence point theorem for hybrid mappings are proved. New types of contractive conditions are used. In [18], the authors have claimed that many fixed point results in G-metric spaces can be deduced from fixed point results in metric spaces, but in our case none of the techniques used in [18] can be applied to our results. Moreover we also present some non-trivial examples and an application for existence of solution of differential inclusion used in heat conduction problems.

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